The Burkill-Cesari Integral for Riesz spaces *

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SUNTO. Si definisce un integrale del tipo "Burkill-Cesari" per funzioni d'insieme a valori in spazi di Riesz Dedekind completi. Si introduce un concetto di quasi-additività, simile a quello introdotto da L. Cesari in [5]. Si provano alcuni teoremi analoghi a quelli classici, e si confronta l'integrale introdotto con quello di Riemann e con quello monotono di cui in [1].

SUMMARY. A definition of "Burkill-Cesari type integral" is given, for set functions, with values in Dedekind complete Riesz spaces. A concept of quasi-additivity is introduced, similar to the one introduced by L. Cesari in [5]. Some theorems analogous to the classical ones are proved. Moreover, we give a comparison with the "Riemann-integral" and the "monotone integral" defined in [1].

1 Introduction.

In 1962 ([5]), L. Cesari gave a definition of integral for set functions, with values in a vector space of finite dimension (the *Burkill-Cesari integral*) and introduced the concepts of quasi-additivity and quasi-subadditivity. He proved that several classical integrals can be viewed as particular cases of this integral. Subsequently, Warner ([11]) extended this integral to the case of set functions with values in a locally convex topological vector space (lctvs). Several authors investigated this type of integration and its related topics: we mention here [9], [10], [3].
Recently, in [7] a theory of integration was developped for real-valued functions, with respect to finitely additive measures, taking values in a lctvs. Moreover, it was proved that this integral can be interpreted as the Burkill-Cesari integral of a suitable set function. Furthermore, in [4]

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a "Riemann-Stieltjes"-type integral was investigated for Dedekind complete Riesz-space-valued set functions.

In this paper, we introduce a "Burkill-Cesari"-type integral for set functions, taking values in a Dedekind complete Riesz space R, and a concept of quasi-additivity and quasi-subadditivity, similar to the ones in [5]. Moreover, we prove some "main" theorems for this type of integral, similar to the classical ones of Cesari ([5]) and Breckenridge ([3]). In particular we prove that, if we introduce a "natural mesh" for a suitable class of intervals, then a bounded R-valued function f, defined in $[a, b]$, is "(R)-integrable" (see [1]) if and only if the corresponding "Mengoli-Cauchy" interval function

$$\eta([\alpha, \beta[) \equiv f(z)(\beta - \alpha),$$

where z is an arbitrary point of $[\alpha, \beta]$, is quasi additive (and hence (BC)-integrable), and **that in this case** the two involved integrals coincide.

In [1], we introduced a "monotone-type" integral for real-valued functions, defined on an arbitrary set X, and with respect to finitely additive R-valued means μ.

In this paper, we shall prove that f is integrable (in the monotone sense) if and only if the "Mengoli-Cauchy" interval function associated with the map

$$u(t) \equiv \mu(\{x \in X : f(x) > t\}), \ t \in \mathbb{R}_0^+,$$

is quasi-additive, and therefore (BC)-integrable, and the two integrals coincide.

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2 Preliminaries.

A Riesz space R is called *Archimedean* if the following property holds: for every choice of $a, b \in R$, $na \leq b$ for all $n \in \mathbb{N}$, implies that $a \leq 0$.

A Riesz space R is said to be *Dedekind complete* [resp. *σ-Dedekind complete*] if every nonempty [countable] subset of R, bounded from above, has least upper bound in R. Every σ-Dedekind complete Riesz space is Archimedean.

Definition 2.1 A directed net $(r_\alpha)_{\alpha \in \Xi}$ is said to be (o)-convergent to r, if

$$(o) - \limsup_{\alpha} \ r_\alpha \equiv \inf_{\alpha} \ \sup_{\beta \geq \alpha} \ r_\beta = (o) - \liminf_{\alpha} \ r_\alpha \equiv \sup_{\alpha} \ \inf_{\beta \geq \alpha} \ r_\beta$$

and we will write $(o) - \lim_\alpha \ r_\alpha = r$.

Definition 2.2 Given an element $r \in R$, we define $r^+ \equiv r \vee 0, \quad r^- \equiv (-r) \vee 0, \; |r| \equiv r \vee (-r)$.

Definition 2.3 A directed net $(r_\alpha)_\alpha$ is said to be (o)-*Cauchy* if

$$(o) - \limsup_{(\alpha,\beta)} |r_\alpha - r_\beta| = 0$$

(see also [8]).

Definition 2.4 Given a fixed element $\xi \in \Xi$, we indicate with the symbol $(o) - \limsup_{\alpha \geq \xi} \; r_\alpha$ [resp. $(o) - \liminf_{\alpha \geq \xi} \; r_\alpha$] the quantity

$$\inf_{\alpha \geq \xi} \sup_{\beta \geq \alpha} r_\beta \;\; [\sup_{\alpha \geq \xi} \inf_{\beta \geq \alpha} r_\beta .]$$

3 The Burkill-Cesari integral.

We now introduce a Burkill-Cesari-type integral for set functions, with values in a Dedekind complete Riesz space R.

Definition 3.1 Let X be any nonempty set, $\mathcal{A}$ an arbitrary nonempty subset of $\mathcal{P}(X)$, R a Dedekind complete Riesz space, $\mathcal{D} \equiv \{D\}$ a directed net of collections of pairwise disjoint subsets of X, belonging to $\mathcal{A}$. Let $\eta : \mathcal{A} \to R$ be a set function, and for all $D \in \mathcal{D}$, define $S(\eta, D) \equiv \sum_{I \in D} \; \eta(I)$. We say that η is *Burkill-Cesari integrable* ((BC)-*integrable*) if there exists in R the limit

$$(o) - \lim_{D \in \mathcal{D}} \; S(\eta, D).$$

When this limit exists, we denote it by the symbol $(BC) - \int_X \; \eta$.

It is easy to prove that, if $\alpha, \; \beta \in I\!R$ and η_1 and η_2 are (BC)-integrable, then $\alpha \; \eta_1 + \beta \; \eta_2$ is (BC)-integrable too, and

$$\int_X \alpha \; \eta_1 + \beta \; \eta_2 = \alpha \int_X \eta_1 + \beta \int_X \eta_2 \; .$$

Definition 3.2 We say that $\eta : \mathcal{A} \to R$ is *quasi-additive* if

$$(o) - \lim_{D_0 \in \mathcal{D}} \; (o) - \limsup_{D \geq D_0} \sum_{I \in D_0} | \sum_{J \in D, \; J \subset I} \eta(J) - \eta(I)| = 0$$

and

$$(o) - \lim_{D_0 \in \mathcal{D}} \; (o) - \limsup_{D \geq D_0} \sum_{J \in D; J \not\subset I, \; \forall \; I \in D_0} |\eta(J)| = 0.$$

The proof of the following proposition is straightforward.

Proposition 3.3 If $\eta_1,\ \eta_2$ are quasi-additive and $\alpha,\ \beta$ are two arbitrary real numbers, then $\alpha\ \eta_1 + \beta\ \eta_2$ **is** quasi-additive.

It is easy to check that, if $R = I\!R$, and there exists a "mesh" $\delta : \mathcal{D} \to I\!R^+$, such that, for every $D_1,\ D_2 \in \mathcal{D},\ [D_1 \geq D_2]$ iff $[\delta(D_1) \leq \delta(D_2)]$, then Definition 3.2 is essentially equivalent to the famous definition of quasi-additivity, proposed by Cesari in [5]:

$\forall\ \varepsilon > 0,\ \exists\ \sigma = \sigma(\varepsilon) > 0$, such that, for every $D_0 \in \mathcal{D}$ with $\delta(D_0) < \sigma$, there exists $\lambda(\varepsilon, D_0) > 0$ such that, for each $D \in \mathcal{D}$ with $\delta(D) < \lambda$, we have:

$$\sum_{I \in D_0} \Big| \sum_{J \in D,\ J \subset I} \eta(J) - \eta(I) \Big| < \varepsilon$$

and

$$\sum_{J \in D; J \not\subset I,\ \forall\ I \in D_0} |\eta(J)| < \varepsilon.$$

The following result holds:

Theorem 3.4 *If η is quasi-additive, then η is (BC)-integrable.*

Proof: We observe that there exists $(p_D)_D,\ p_D \downarrow 0$, such that, for all $D_0, D_1, D_2 \in \mathcal{D}$, with $D_1 \geq D_0,\ D_2 \geq D_0$, one has:

$$\begin{aligned} & (o) - \limsup_{(D_1,D_2)} |S(\eta, D_1) - S(\eta, D_2)| = (o) - \limsup_{(D_1,D_2), D_1 \geq D_0, D_2 \geq D_0} |S(\eta, D_1) - S(\eta, D_2)| \leq \\ \leq\ & (o) - \limsup_{D_1 \geq D_0} \sum_{I \in D_0} \Big| \sum_{J \in D_1, J \subset I} \eta(J) - \eta(I) \Big| + (o) - \limsup_{D_1 \geq D_0} \sum_{J \in D_1; J \not\subset I,\ \forall\ I \in D_0} |\eta(J)| + \\ +\ & (o) - \limsup_{D_2 \geq D_0} \sum_{I \in D_0} \Big| \sum_{J \in D_2, J \subset I} \eta(J) - \eta(I) \Big| + (o) - \limsup_{D_2 \geq D_0} \sum_{J \in D_2; J \not\subset I,\ \forall\ I \in D_0} |\eta(J)| \leq p_{D_0}. \end{aligned}$$

By arbitrariness of $D_0 \in \mathcal{D}$, we get:

$$(o) - \limsup_{(D_1,D_2)} |S(\eta, D_1) - S(\eta, D_2)| = 0.$$

So, the net $\{S(\eta, D)\}_{D \in \mathcal{D}}$ is Cauchy, and hence it is convergent, by virtue of Dedekind completeness of R (see also [8]).

Definition 3.5 We say that η is *quasi-subadditive* if

$$(o) - \lim_{D_0 \in \mathcal{D}} (o) - \limsup_{D \geq D_0} \sum_{I \in D_0} \Big[\sum_{J \in D, J \subset I} \eta(J) - \eta(I) \Big]^- = 0.$$

It is readily seen that, if $\alpha,\ \beta \in I\!R_0^+$ and $\eta_1,\ \eta_2$ are quasi-subadditive, then $\alpha\ \eta_1 +\ \beta\ \eta_2$ is quasi-subadditive too: indeed, it is enough to recall that

$$(a+b)^- \le a^- + b^-;\ (\alpha\ a)^- = \alpha\ a^-\ ,$$

$\forall\ a, b \in R$ and $\alpha \in I\!R_0^+$ (see also [6]).

Theorem 3.6 *Let η be positive, quasi-subadditive and such that*

$$(o) - \limsup_{D \in \mathcal{D}}\ S(\eta, D)$$

exists in R. Then, η is quasi-additive.

Proof: First of all, we prove (BC)-integrability of η. Let $D \ge D_0 \in \mathcal{D}$. We have:

$$\begin{aligned} S(\eta, D) - S(\eta, D_0) &= \sum_{J \in D} \eta(J) - \sum_{I \in D_0} \eta(I) = \sum_{I \in D_0} [\sum_{J \in D,\ J \subset I} \eta(J) - \eta(I)] + \sum_{J \in D; J \not\subset I,\ \forall\ I \in D_0} \eta(J) \ge \\ &\ge \sum_{I \in D_0} [\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)] \ge - \sum_{I \in D_0} [\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)]^- \ge -p_{D_0}, \end{aligned}$$

where $p_{D_0} \downarrow 0$ (indeed, $a \ge -a^-,\ \forall\ a \in R$), and hence

$$l^{\ (1)} \ge S(\eta, D_0) - p_{D_0}\ ,\ \ \forall\ D_0 \in \mathcal{D},$$

where $l^{\ (1)} = (o) - \liminf_{D \in \mathcal{D}}\ S(\eta, D)$. From this, it follows that

$$(o) - \limsup_{D_0 \in \mathcal{D}}\ S(\eta, D_0) \le l^{\ (1)} + (o) - \limsup_{D_0 \in \mathcal{D}}\ p_{D_0} = l^{\ (1)}.$$

So, there exists in R the quantity $l \equiv (o) - \lim_{D \in \mathcal{D}}\ S(\eta, D)$, and thus η is (BC)-integrable.

Now we shall use the following equalities: $|a| = a^+ + a^-,\ a = a^+ - a^-$, and hence $|a| = a + 2\ a^-$. Pick arbitrarily $D,\ D_0 \in \mathcal{D}$, with $D \ge D_0$. We have:

$$\begin{aligned} 0 &\le \sum_{I \in D_0} |\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)| + \sum_{J \in D;\ J \not\subset I,\ \forall\ I \in D_0} \eta(J) = \\ &= \sum_{I \in D_0} [\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)] + 2 \sum_{I \in D_0} [\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)]^- + \sum_{J \in D;\ J \not\subset I,\ \forall\ I \in D_0} \eta(J) \le \\ &\le |\sum_{J \in D} \eta(J) - l| + |\sum_{I \in D_0} \eta(I)\ - l| + 2 \sum_{I \in D_0} [\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)]^- \le 2\ p_{D_0} + 2\ q_{D_0}\ , \end{aligned}$$

for some suitable nets $(p_D)_D,\ (q_D)_D$ in R, with $p_D \downarrow 0,\ q_D \downarrow 0$. Taking the $(o) - \limsup$, we get:

$$0 \le (o) - \limsup_{D \ge D_0} \sum_{I \in D_0} |\sum_{J \in D,\ J \subset I} \eta(J)\ - \eta(I)|,\ (o) - \limsup_{D \ge D_0} \sum_{J \in D;\ J \not\subset I,\ \forall\ I \in D_0} |\eta(J)| \le$$

$$\leq (o) - \limsup_{D \geq D_0} \sum_{I \in D_0} | \sum_{J \in D,\ J \subset I} \eta(J) \ - \eta(I)| + (o) - \limsup_{D \geq D_0} \sum_{J \in D;\ J \not\subset I,\ \forall\ I \in D_0} |\eta(J)| \leq$$

$$\leq 2\ p_{D_0} + 2\ q_{D_0}.$$

Thus, it follows that η is quasi-additive, that is the assertion. □

Definition 3.7 Given a set function $\eta : \mathcal{A} \to R$, define $\eta^+,\ \eta^-,\ |\eta| : \mathcal{A} \to R$ as follows:

$$\eta^+(I) \equiv [\eta(I)]^+,\ \eta^-(I) \equiv [\eta(I)]^-,\ |\eta|(I) \equiv |\eta(I)|\ , \forall\ I \in \mathcal{A}.$$

Theorem 3.8 *If η is quasi-additive, then $\eta^+,\ \eta^-$ and $|\eta|$ are quasi-subadditive.*

The proof is analogous to the one given in [5].

Definition 3.9 Under the same notations as above, let $M \subset X$, and define $S(\eta, M, D) \equiv \sum_{I \in D}\ s(I,M)\ \eta(I)$, where:

$$s(I,M) \equiv \begin{cases} 1, \text{ if } I \subset M \\ \\ 0, \text{ if } I \not\subset M. \end{cases}$$

We say that η is *Burkill-Cesari integrable* ((BC)-*integrable*) on M if there exists in R the limit

$$(o) - \lim_{D \in \mathcal{D}}\ S(\eta, M, D).$$

When this limit exists, we denote it by the symbols $(BC) - \int_X\ [\eta, M]$ or $(BC) - \int_M\ \eta$.
The set function $\eta : \mathcal{A} \to R$ is *quasi-additive on M* if

$$(o) - \lim_{D_0 \in \mathcal{D}}\ (o) - \limsup_{D \geq D_0} \sum_{I \in D_0} s(I,M)\ |\eta(I) - \sum_{J \in D} s(J,I)\ \eta(I)| = 0$$

and

$$(o) - \lim_{D_0 \in \mathcal{D}}\ (o) - \limsup_{D \geq D_0} \sum_{J \in D} s(J,M)\ [1 - \sum_{I \in D_0} s(J,I)\ s(I,M)]|\eta(J)| = 0\ .$$

We say that η is *quasi-subadditive on M* if

$$(o) - \lim_{D_0 \in \mathcal{D}}\ (o) - \limsup_{D \geq D_0} \sum_{I \in D_0} s(I,M)\ [\sum_{J \in D} s(J,I)\ \eta(J) - \eta(I)]^- = 0\ .$$

It is easy to check that, if η is quasi-subadditive, then it is quasi-subadditive on each set $M \in \mathcal{A}$.

Theorem 3.10 *If η is quasi additive, and $\int_X \ |\eta|$ exists in R, then η is quasi additive on every set $M \in \mathcal{A}$.*

Proof: Let $M \in \mathcal{A}$. By Theorem 3.8, $|\eta|,\ \eta^+,\ \eta^-$ are positive and quasi subadditive, and so they are quasi subadditive on M. So,

$$0 \leq \int_M \ \eta^+ \ , \ \int_M \ \eta^- \leq \int_M \ |\eta| \leq \int_X \ |\eta|$$

exist in R, and hence $|\eta|,\ \eta^+,\ \eta^-$ are quasi-additive on M, by reasoning as in Theorem 3.6. Thus, $\eta = \eta^+ - \eta^-$ is quasi-additive on M, that is the assertion. □

4 Integrals of Riesz-space-valued functions with respect to real-valued measures

Now we compare the introduced Burkill-Cesari-type integral with other integrals, existing in the literature.

Let R be a Dedekind complete Riesz space, $u : [a, b] \to R$ be a bounded map. In [1], we defined a Riemann - type integral, which can be defined equivalently as a "Mengoli-Cauchy" type integral.

Definition 4.1 Given an interval $[a, b] \subset I\!\!R$, we call *division of* $[a, b]$ any finite set $\{x_0,\ x_1, \dots,\ x_n\} \subset [a, b]$, where $x_0 = a,\ x_n = b$, and $x_i < x_{i+1},\ \forall\ i = 0, \dots, n$. We denote by $\mathcal{D}$ the class of all divisions of $[a, b]$.

We call *mesh* of a division D the quantity $\delta(D) \equiv \max_i \ (x_i - x_{i-1})$, and say that $D_1 \geq D_2$ if $\delta(D_1) \leq \delta(D_2)$.

A division D is identified with the collection of intervals $[x_{i-1}, x_i|$, where

$$[\alpha, \beta| \equiv \begin{cases} [\alpha, \beta[& \text{if} \beta \neq b \\ \\ [\alpha, \beta] & \text{if} \beta = b. \end{cases}$$

We now recall some definitions of integral given in [1].

Definition 4.2 Let R be a Dedekind complete Riesz space, and $u : [a, b] \to R$ a bounded map. We say that a map $g : [a, b] \to R$ is a *step function* with respect to $\mathcal{D}$ if there exist $n + 1$ points $x_0 \equiv a < x_1 < \dots < x_n \equiv b$, such that g is constant in each interval of the type $]x_{i-1}, x_i[$ $(i = 1, \dots, n)$. If g is a step function, we put $\int_a^b \ g(t)\ dt \equiv \sum_{i=1}^n \ (x_i - x_{i-1}) \cdot g(\xi_i)$ where ξ_i is an arbitrary point of $]x_{i-1}, x_i[$.

We call *upper integral* [resp. *lower integral*] of u the element of R given by

$$\inf_{v \in V_u} \int_a^b v(t)\ dt \ \ [\sup_{s \in S_u} \int_a^b s(t)\ dt],$$

where

$$V_u \equiv \{v : v \text{ is a step function },\ v(t) \geq u(t),\ \forall\ t \in [a,b]\}$$
$$S_u \equiv \{s : s \text{ is a step function },\ s(t) \leq u(t),\ \forall\ t \in [a,b]\}.$$

We say that a bounded function $u : [a,b] \to R$ is *Riemann* integrable (or (R)-*integrable*), if its lower integral coincides with its upper integral, and, in this case, we call *integral of* u (and write $\int_a^b u(t)\ dt$) their common value, and we indicate it by

$$(R) - \int_a^b u(t)\ dt.$$

Definition 4.3 Let $[a,b] \subset I\!R, R$ be as above, and $u : [a,b] \to R$ be a map. We say that u is *Mengoli-Cauchy integrable* ((MC)-*integrable*) if there exists an element $I \in R$ such that

$$(o) - \lim_{D \in \mathcal{D}} |\sum_{i=1}^n u(z_i)(x_i - x_{i-1}) - I| = 0,$$

uniformly with respect to $z_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$, and we write $(MC) - \int_a^b u(t)\ dt \equiv I$.

Every Mengoli-Cauchy integrable function is bounded. The following results hold (see also [2]):

Theorem 4.4 *Let $u : [a,b] \to R$ be Mengoli-Cauchy integrable. Then, u is bounded and Riemann integrable, and*

$$(R) - \int_a^b u(t)\ dt = (MC) - \int_a^b u(t)\ dt.$$

Theorem 4.5 *Let $u : [a,b] \to R$ be Riemann integrable. Then, u is Mengoli-Cauchy integrable, and*

$$(MC) - \int_a^b u(t)\ dt = (R) - \int_a^b u(t)\ dt.$$

Definition 4.6 A map $u : [a,b] \to R$ is called *continuous* at the point $x_0 \in [a,b]$ if

$$(o) - \lim_{x \to x_0} u(x) = u(x_0).$$

A function $u : [a,b] \to R$ is said to be *differentiable* at x_0 if

$$(o) - \lim_{x \to x_0} \frac{u(x) - u(x_0)}{x - x_0} \ \text{ exists in } R.$$

Remark 4.7 We note that there exist Riemann integrable functions $u : [a, b] \to R$, which are discontinuous at every $x \in]a, b[$.
Indeed, let $[a, b] \equiv [0, 1]$, $R \equiv I\!R^{[0,1]}$, $u(s) \equiv \chi_{[0,s]}$, $\forall\ s \in [0, 1]$. For each $x \in]0, 1[$, we have:

$$\lim_{t\to x^+} u(t) = \chi_{[0,x]}, \qquad \lim_{t\to x^-} u(t) = \chi_{[0,x[},$$

and hence

$$\limsup_{t\to x} u(t) - \liminf_{t\to x} u(t) = \chi_{\{x\}} \not\leq \frac{1}{2}.$$

However, u is Riemann integrable. Put $I(s) \equiv (R) - \int_0^s u(t)\ dt$. It is easy to check that

$$I(s)(x) = \begin{cases} 0 & \text{if } x \geq s \\ \\ s - x & \text{if } x < s \end{cases}$$

with $\forall\ s, x \in [0, 1]$, and that the "right derivative" of $I(s)$ is $u(s)$, $\forall\ s \in [0, 1]$.
Moreover, it is easy to prove that, if $u : [a, b] \to R$ is an (R)-integrable function, then the map $I(s) \equiv (R) - \int_0^s u(t)\ dt$ is differentiable at the points s for which u is continuous, and in such points $I'(s) = u(s)$.

Now, let $\mathcal{A}$ be the collection of all subintervals of $[a, b]$ of the type $[\alpha, \beta|$, and set $\eta([\alpha, \beta|) \equiv u(z)\ (\beta - \alpha)$, where z is an arbitrary point of $[\alpha, \beta]$. Obviously, a bounded function $u \in R^{[a,b]}$ is (MC)-integrable if and only if η is (BC)-integrable.
We now prove the following:

Theorem 4.8 *If u is (R)-integrable, then η is quasi-additive.*

Proof: Without loss of generality, we may assume that u is positive. Indeed, if u is (R)-integrable, then u^+ and u^- are (R)-integrable too.
As u is bounded, $(o) - \limsup_{D\in\mathcal{D}} S(\eta, D)$ exists in R. So, it will be enough to show that η is quasi-subadditive, in view of Theorem 3.6.
Let $D_0 \equiv \{[c_{i-1}, c_i| : i = 1, \ldots, N{-}1\}$, $D \equiv \{[x_{j-1}, x_j| : j = 1, \ldots, n\}$, where $c_0 = a < c_1 < \ldots < c_{N-1} = b$, $x_0 = a < x_1 < \ldots < x_n = b$, $\delta(D) \leq \delta(D_0)$. Moreover, set

$$M \equiv \sup_{x\in[a,b]} u(x); \; M_i \equiv \sup_{x\in[c_{i-1},\ c_i]} u(x),$$

$$m_i \equiv \inf_{x\in[c_{i-1},\ c_i]} u(x).$$

By virtue of (R)-integrability of u, we have:

$$\sum_{I\in D_0} |\sum_{J\in D,\ J\subset I} \eta(J)-\eta(I)| \le \sum_{I\in D_0} (M_i-m_i)(c_i-c_{i-1})+M \sum_{J\in D,\ J\not\subset I,\ \forall\ I\in D_0} (x_j-x_{j-1}) \le$$

$$\le p_D + N\ \delta(D)\ M,$$

for some suitable directed net $(p_D)_D,\ p_D \downarrow 0$. (We note that $N=N(D_0)$ depends on D_0.) So,

$$0 \le (o)-\limsup_{D\ge D_0} \sum_{I\in D_0} |\sum_{J\in D,\ J\subset I} \eta(J)-\eta(I)| \le p_{D_0} + \inf_{D\ge D_0} N(D_0)\ \delta(D)\ M = p_{D_0},$$

and hence

$$(o)-\lim_{D_0\in\mathcal{D}} (o)-\limsup_{D\ge D_0} \sum_{I\in D_0} |\sum_{J\in D, J\subset I} \eta(J)-\eta(I)| = 0.\ \square$$

Next, we show that quasi-additivity can be applied in a different problem.

Definition 4.9 A map $g:[a,b]\to R$ is said to be *of bounded variation* if the set

$$\{\sum_{I\in D} |q_g(I)| : D\in\mathcal{D}\}$$

is bounded in R, where

$$q_g([u,v|) \equiv g(v)-g(u).$$

In this case, we set

$$V(g,[a,b]) \equiv \sup\{\sum_{I\in D} |q_g(I)| : D\in\mathcal{D}\}.$$

The following result holds.

Theorem 4.10 *If $g:[a,b]\to R$ is of bounded variation and continuous in $[a,b]$, then the function $|q_g|$ is quasi-additive.*

Proof: We observe that, in order to prove Theorem 4.10, it is enough to prove quasi-subadditivity of $|q_g|$. Indeed, quasi-additivity will follow from Theorem 3.6.
Fix $D_0\in\mathcal{D},\ D_0\equiv\{[c_{i-1},c_i| : i=1,\dots,N-1\}$. By the continuity of g at the points $c_i,\ i=1,\dots,N-1$, there exists a sequence $(p_n(c_i))_n,\ p_n\downarrow 0$, such that

$$|q_g([u,v])| \le p_n, \text{whenever } a\le u\le c_i\le v\le b,\ 0\le v-u\le \frac{1}{n}\ \ (i=1,\dots,N)$$

Let $D \in \mathcal{D}$, $D \equiv \{[x_{j-1}, x_j| : j = 1, \dots, k\}$. with $\delta(D) \leq \frac{1}{n}$.
Put $E \equiv \{I \in D_0 : \exists\, j, x_j \in I\}$ (Note that if $[\alpha, \beta| \in D_0 \setminus E$, then $\beta - \alpha \leq \frac{1}{n}$). For $I_i = [c_{i-1},\ c_i| \in E$, define $d_i \equiv \min\{x_j : x_j \geq c_{i-1}\}$ and $e_i \equiv \max\{x_j : x_j < c_i\}$. Then

$$\sum_{I \in D_0} \Big(\sum_{J \in D, J \subset I} |q_g(J)| - |q_g(I)|\Big) \geq \sum_{I_i \in E} (|q_g([d_i, e_i|)| - |q_g(I_i)|) + \sum_{I_i \in D_0 \setminus E} - |q_g(I_i)| \geq$$

$$\geq \sum_{I_i \in E} - (|g(d_i) - g(c_{i-1})| + |g(e_i) - g(c_i)|] + \sum_{I_i \in D_0 \setminus E} (-p_n(c_i)).$$

So, $\forall\ D_0 \in \mathcal{D}$,

$$(o) - \limsup_{D \geq D_0} \sum_{I \in D_0} \Big[\sum_{J \in D, J \subset I} |q_g(J)| - |q_g(I)|\Big]^- = 0.$$

Thus, $|q_g|$ is quasi-subadditive. □

We note that $\int_a^b |q_g| = V(g, [a, b])$ (see also [4]).
Now, we recall the integral for extended real-valued functions, with respect to R-valued means, defined in [1].

Definition 4.11 Let X be any set, $\mathcal{B} \subset \mathcal{P}(X)$ be an algebra, R be a Dedekind complete Riesz space, $\mu : \mathcal{B} \to R$ be a finitely additive positive set function; assume that $f : X \to \mathbb{R}_0^+$ is a measurable function, and $u(t) \equiv \mu(\{x \in X : f(x) > t\})$. We say that f is *integrable* if there exists in R the quantity

(4.11.1) $\displaystyle\int_0^{+\infty} u(t)\ dt \equiv \sup_{a > 0} \int_0^a u(t)\ dt = (o) - \lim_{a \to +\infty} \int_0^a u(t)\ dt,$

where the integral in (4.11.1) is intended as in Definition 4.2. If f is integrable, we indicate the element in (4.11.1) by the symbol $\int_X f\ d\mu$.
A measurable function $f : X \to \mathbb{R}$ is *integrable* if both f^+, f^- are integrable and, in this case, we set

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

Remark 4.12 It is easy to check that, if $f : X \to \mathbb{R}^+$ is integrable (in the monotone sense), then

$$\int_X f\ d\mu = \sup_{\mathcal{D}} \sum_{i=1}^n u(x_i)(x_i - x_{i-1}) = \inf_{\mathcal{D}} \sum_{i=1}^n u(x_{i-1})(x_i - x_{i-1}),$$

where $\mathcal{D}$ is the class of all finite subsets of $[0, +\infty[$ of the type $\{x_0 = 0, x_1, \dots, x_n\}$, $n \in \mathbb{N}$, by virtue of (decreasing) monotonicity of u.

Now, let $\mathcal{A} \equiv \{[a,b[\colon a,b \in I\!\!R_0^+,\ a<b\};\ \eta(I) \equiv u(x_{i-1})\ (x_i - x_{i-1}),\ \delta(D) \equiv \max_{i=1}^{n-1}\ (x_i - x_{i-1}) + \frac{1}{x_n}$, $\forall\ D \in \mathcal{D}$. By proceeding analogously as in the previous case, and by virtue of the properties of the function u, one can prove that a nonnegative function $f \in I\!\!R^X$ is integrable (in the monotone sense) if and only if η is quasi-additive, and the (BC)-integral of η coincides with $\int_X\ f\ d\mu$.

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