The Monotone Integral with respect to Riesz space-valued Capacities

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SUNTO. Si introduce un "integrale monotono" (nello stesso spirito di [5]), per applicazioni a valori reali e rispetto a funzioni d'insieme monotone non decrescenti e a valori in spazi di Riesz Dedekind completi. Si dimostrano teoremi di rappresentazione (tra cui una versione del Teorema di Rappresentazione di Riesz). Inoltre si introduce una nozione di convergenza debole, e vengono provati teoremi tipo Portmanteau, Vitali e Fatou. Inoltre è dimostrata una versione della legge forte e della legge debole dei grandi numeri.

SUMMARY. A definition of "monotone integral" is given, similarly as in [5], for real-valued maps and with respect to Dedekind complete Riesz space-valued "capacities". Some representation theorems are proved; in particular, we give here a version of Riesz representation theorem. Moreover, a concept of weak convergence is introduced, and some Portmanteau-type theorems, Vitali convergence and Fatou theorems are proved. Finally, a version of both strong and weak laws of large numbers is demonstrated.

1 Introduction.

In the literature, in certains types of studies (for example, in stochastic processes), it would be "natural" to investigate some kinds of "probabilities", which to every event associate not simply a real number, but a real-valued function.
Indeed, one can give different valuations of the uncertainty of some event E, depending, for example, on the "informations", which one can receive, during his study about E or about some other events, "related" to E.
For example, given a measurable space (X, Σ), we can consider applications $\mathcal{P} : \Sigma \to [0,1]^T$ in order to stress that the "probability" of each event A depends on the "time": $P(A)$ is a function of $t \in T$.
As a second example, given $\mathcal{Z}$ a sub-σ-algebra of Σ and a probability P on (X, Σ), we can define the "conditional probability" as follows: $\widetilde{P}(A) = P(A|\mathcal{Z}) = E(1_A|\mathcal{Z})$ for every $A \in \Sigma$. So $\widetilde{P} : \Sigma \to L_0$.
More generally, it would be advisable to associate to each event an element of a Riesz space R : indeed, we note that, thanks to Maeda-Ogasawara-Vulikh representation Theorem, every Archimedean Riesz space can

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be viewed as a suitable space of continuous extended real-valued functions.
On the other hand, in the literature there exist several contributions to the foundations of "qualitative probabilities " and their "realizations", which can be represented not only by additive functions but also by submodular capacities (see [7], [11], [12]). So, it will be natural, in certains problems, to consider "probabilities", as just monotonic functions, with values in Riesz spaces.
As a further example, we can consider stochastic integration, when we define the integral of a scalar function with respect to a stochastic measure I_X, where $X : \Omega \times I\!R^+ \to L^p$ is a process.
Another motivation for the study of the integral with respect to capacities is that, in the theory of decisions, the "preference" relations between "measurable" functions, which are the "acts" of the considered individual, can be represented by means of the Choquet integral of some suitable utility functions. In particular, if X denotes the space of all "choices" (i.e., states of nature) and Γ is the space of all possible "consequences", an *act* is a "measurable" mapping $f : X \to \Gamma$. In uncertainty conditions, to state that "f is preferable to g" is equivalent to say that it is possible to determine a capacity P and a utility function $u : \Gamma \to I\!R$, in such a way that

$$\int_X u \circ f \; dP \geq \int_X u \circ g \; dP.$$

In particular, if we operate under risk condition, P must be a measure (see [10]).
In [5], we introduced a "monotone-type" integral for real-valued functions, with respect to finitely additive positive set functions, with values in a Dedekind complete Riesz space. In Section 3, we define a "monotone integral" for real-valued maps, with respect to monotone Riesz-space-valued set functions, and we study his properties. Among "similar" integrals existing in the literature, we recall the "monotone integral" of real-valued functions with respect to measures with values in a Banach space (see [6]), in a locally convex vector topological space (see [17]), and the "fuzzy-type" integral for a lattice-valued function with respect to lattice-valued measures (see [19]). For this integral, we prove some types of representation theorems (for similar theorems existing in the literature for the real case, see [9], [10], [13], [16], [21], [22]), and in Theorem 3.17. we extend a result of Schmeidler (see [9], [23]); and so to integrate comonotonic functions with respect to capacities is equivalent to integrating them with respect to suitable measures.
In Section 4, we prove a version of Riesz representation theorem. More precisely, given a normal topological space X, a Dedekind complete Riesz space R and a linear monotone R-valued functional T, we construct an R-valued set function μ, which is monotone on $\mathcal{P}(X)$, satisfies some properties of "regularity", and is finitely additive on the algebra $\mathcal{M}$ generated by all open sets and such that $T(f) = \int_X f d\mu$, for each $f \in C_b(X) = \{f : X \to I\!R, f \text{ continuous and bounded } \}$. Moreover, μ is σ-additive on $\mathcal{M}$, in the case in which X is compact.
However, in general, it is impossible to obtain the existence of a set function μ, which is additive on the Borel σ-field, even if X is compact and Hausdorff (see also [24]). In general, this is possible if R is *weakly σ-distributive* ; in fact, a Riesz space is weakly σ-distributive if and only if every σ-additive set function, defined in any algebra $\mathcal{M}$, has a σ-additive extension, defined on the smallest σ-algebra containing $\mathcal{M}$ (see [25]). Furthermore, we introduce a definition of weak convergence for Riesz space-valued capacities, and prove some versions of Portmanteau, Vitali and Fatou 's theorem, with respect to this kind of convergence.
Finally, we prove a version of both strong and weak laws of large numbers for the introduced integral, with respect to σ-additive and finitely additive R-valued set functions respectively.

2 Preliminaries.

A Riesz space R is called *Archimedean* if the following property holds: for every choice of $a, b \in R$, $na \le b$ for all $n \in I\!\!N$, implies that $a \le 0$.
A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

Throughout this paper, we always suppose that R is a Dedekind complete Riesz space.

Proposition 2.1 [1] *Every Dedekind complete Riesz space is Archimedean.*

Theorem 2.2 [2] *Given a Dedekind complete Riesz space R, there exists a compact Stonian topological space Ω, unique up to homeomorphisms, such that R can be embedded as a solid subspace of $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{I\!\!R}^{\Omega}$: f is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda\in\Lambda}$ is any family such that $a_\lambda \in R \;\forall\; \lambda$, and $a = \sup_\lambda\; a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \sup_\lambda\; a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$, and the set $\{\omega \in \Omega : (\sup_\lambda\; a_\lambda)(\omega) \neq \sup_\lambda\; a_\lambda(\omega)\}$ is meager in Ω.*

Definition 2.3 A sequence $(r_n)_n$ is said to be (o)-convergent to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \le p_n, \;\forall\; n \in I\!\!N$, and we will write $(o) - \lim_n\; r_n = r$.

Definition 2.4 A sequence $(r_n)_n$ is said to be (o)-*Cauchy* if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r_m| \le p_n, \;\forall\; n \in I\!\!N$, and $\forall\; m \ge n$.

Definition 2.5 Let R be a Riesz space, I be a connected subset of $I\!\!R$. We say that $w : I \to R$ is [*right, left*] *continuous at a fixed point* $x_0 \in I$ if

$$(o) - \lim_{x \to x_0[^+][^-]} w(x) = w(x_0).$$

The map f is called [*right, left*] *continuous* if it is [right, left] continuous at every point $x_0 \in I$.

Definition 2.6 If X is any topological space, we indicate by the symbol $\mathcal{C}_b(X)$ the class of all continuous bounded real-valued functions, where $I\!\!R$ is endowed with the usual topology.

Definition 2.7 Let X be an arbitrary set, and $f \in I\!\!R^X$. The class

$$\mathcal{Q}_f \equiv \{\; \{x \in X : f(x) > t\} : t \in I\!\!R\} \cup \{\{x \in X : f(x) \ge t\} : t \in I\!\!R\}$$

is called the *upper set system* of f.

Definition 2.8 We say that a class $\mathcal{C}$ of elements of $I\!\!R^X$ is *comonotonic* if $\cup_{f\in\mathcal{C}}\; \mathcal{Q}_f$ is a chain, or equivalently, if, for each pair of $f, g \in \mathcal{C}$, there is no pair of elements $x_1,\; x_2 \in X$, such that $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$ (see [9]).

3 The monotone integral for capacities.

Definition 3.1 Let X be any set, and $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra.
We say that a set function $P : \mathcal{A} \to R$ is a *capacity* if $P(\emptyset) = 0$, and $P(A) \leq P(B)$ whenever $A, B \in \mathcal{A},\ A \subset B$; P is said to be *submodular* if

$$A,\ B \in \mathcal{A} \Longrightarrow P(A \cup B) + P(A \cap B) \leq P(A) + P(B);$$

supermodular , if

$$A,\ B \in \mathcal{A} \Longrightarrow P(A \cup B) + P(A \cap B) \geq P(A) + P(B);$$

subadditive , if

$$A,\ B \in \mathcal{A} \Longrightarrow P(A \cup B) \leq P(A) + P(B);$$

superadditive , if

$$A,\ B \in \mathcal{A} \Longrightarrow P(A \cup B) \geq P(A) + P(B).$$

A map $P : \mathcal{A} \to R$ is said to be a *mean* if $P(A) \geq 0,\ \forall\ A \in \mathcal{A}$, and $P(A \cup B) = P(A) + P(B)$, whenever $A \cap B = \emptyset$. A mean P is σ*-additive* if $\inf_n\ P(A_n) = 0$, whenever $(A_n)_n$ is a decreasing sequence in $\mathcal{A}$, such that $\cap_n\ A_n = \emptyset$, or equivalently if

$$P(\cup_{n=1}^{\infty}\ B_n) = \sum_{n=1}^{\infty} P(B_n),$$

whenever $(B_n)_n$ is any disjoint sequence of elements of $\mathcal{A}$, such that $\cup_{n=1}^{\infty}\ B_n \in \mathcal{A}$.

We say that a set function P is a *measure* if it is a σ-additive mean.

Definition 3.2 Assume that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, $\mathcal{F},\ \mathcal{G} \subset \mathcal{A}$ are two lattices, such that $\emptyset \in \mathcal{F}$, and the complement (with respect to X) of every element of $\mathcal{F}$ belongs to $\mathcal{G}$. A mean P on $\mathcal{A}$ is called *tight* if the following properties hold:

R1) $\forall\ F \in \mathcal{F},\ \forall\ n \in I\!N,\ \exists\ G_n \in \mathcal{G}$ such that $F \subset G_{n+1} \subset G_n\ \forall\ n$, and $\inf_n\ P(G_n \setminus F) = 0$.

Remark 3.3 It is easy to see that, if X is a metric space, $\mathcal{S} = \{$ Borel sets$\}$, $\mathcal{F} = \{$ closed sets $\}$, and $\mathcal{G} = \{$ open sets $\}$, then every σ-additive mean is tight.

Definition 3.4 Let X be a topological space, and assume that $\mathcal{A}$ contains all open subsets of X. We say that a set function $P : \mathcal{A} \to R$ is *regular* (on $\mathcal{A}$) if, for every $E \in \mathcal{A}$,

$$P(A) = \inf\{P(V) : A \subset V,\ V \text{ is open }\}$$

and

$$P(A) = \sup\{P(C) : A \supset C,\ C \text{ is closed }\}$$

Definition 3.5 If $K \subset X$ is closed, $f \in [0,1]^X$ is a continuous function, we say that $K \prec f$ if $f(x) = 1,\ \forall\ x \in K$. If $V \subset X$ is open, we write that $f \prec V$ if its support is contained in V.

The following result holds:

Proposition 3.6 *Let X be a compact topological space, and assume that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, containing the class of all open sets. Then, every regular mean P on $\mathcal{A}$ is σ-additive on $\mathcal{A}$.*

Proof: Let $(A_k)_k$ be a sequence of disjoint sets in $\mathcal{A}$, such that $A \equiv \cup_k A_k \in \mathcal{A}$. It is easy to see that

$$P(A) \geq \sum_{k=1}^{\infty} P(A_k).$$

We now prove the opposite inequality. Fix $\varepsilon > 0$ and $A \in \mathcal{A}$, and let Ω be as in Theorem 2.2. By regularity of P, there exists a meager set J such that, $\forall\, \omega \in \Omega \setminus J$, there exists a closed set $C^\omega \subset A$, such that

$$P(A)(\omega) - P(C^\omega)(\omega) \leq \frac{\varepsilon}{2}$$

and there are open sets $U_k^\omega,\ U_k^\omega \supset A_k\ \forall\, k$, such that

$$P(U_k^\omega)(\omega) - P(A_k)(\omega) \leq \frac{\varepsilon}{2^{k+1}}.$$

As X is compact, then C^ω is too: so, for each $\omega \notin J$, there exists $n(\omega) \in I\!\!N$, such that $C^\omega \subset \cup_{k=1}^{n(\omega)} U_k^\omega$. Thus, $\forall\, \omega \notin J$, we get:

$$\begin{aligned} P(A)(\omega) \quad &\leq \quad P(C^\omega)(\omega) + \frac{\varepsilon}{2} \leq P(\cup_{k=1}^{n(\omega)} U_k^\omega)(\omega) + \frac{\varepsilon}{2} \leq \\ &\leq \quad \sum_{k=1}^{n(\omega)} P(U_k^\omega)(\omega) + \frac{\varepsilon}{2} \leq \sum_{k=1}^{\infty} \left(P(A_k)(\omega) + \frac{\varepsilon}{2^{k+1}}\right) + \frac{\varepsilon}{2} \leq \sum_{k=1}^{\infty} P(A_k)(\omega) + \varepsilon. \end{aligned}$$

As the complement of a meager set is dense, we get:

$$P(A)(\omega) \leq [\sum_{k=1}^{\infty} P(A_k)](\omega), \quad \forall \omega \in \Omega$$

and hence

$$P(A) \leq \sum_{k=1}^{\infty} P(A_k) \quad \Box.$$

As in [5], given a mapping $f : X \to \tilde{I\!\!R}$ and a capacity P, for all $A \in \mathcal{A}$, set: $\Sigma_{t,A}^f$ (or simply $\Sigma_{t,A}$, when no confusion can arise) $\equiv \{x \in A : f(x) > t\}$; Σ_t^f $(\Sigma_t) \equiv \{x \in X : f(x) > t\}$; and, for every $t \in I\!\!R$, let $u_{A,f}(t) \equiv P(\Sigma_{t,A}^f)$; $u_f(t) = u(t) \equiv P(\Sigma_t)$.

Definition 3.7 We say that a map $f : X \to \tilde{I\!\!R}$ is *measurable* if $\Sigma_t^f \in \mathcal{A},\ \forall\, t \in I\!\!R$. A real-valued measurable map is called *random variable* too.

Now, we define a Riemann-type integral, for maps, defined in an interval of the real line, and taking values in a Dedekind complete Riesz space.

Definition 3.8 Let $a, b \in I\!\!R,\ a < b$, and R be as above. We say that a map $g : [a, b] \to R$ is a *step function* if there exist $n + 1$ points $x_0 \equiv a < x_1 < \ldots < x_n \equiv b$, such that g is constant in each interval of the type $]x_{i-1}, x_i[$ $(i = 1, \ldots, n)$. If g is a step function, we put $\int_a^b g(t)\, dt \equiv \sum_{i=1}^n (x_i - x_{i-1}) \cdot g(\xi_i)$, where ξ_i is an arbitrary point of $]x_{i-1}, x_i[$.

Definition 3.9 Let $u : [a, b] \to R$ be a bounded function. We call *upper integral* [resp. *lower integral*] of u the element of R given by

$$\inf_{v \in V_u} \int_a^b v(t)\ dt \ \ [\sup_{s \in S_u} \int_a^b s(t)\ dt],$$

where

$$\begin{aligned} V_u &\equiv \{v : v \text{ is a step function },\ v(t) \geq u(t),\ \forall\ t \in [a, b]\} \\ S_u &\equiv \{s : s \text{ is a step function },\ s(t) \leq u(t),\ \forall\ t \in [a, b]\}. \end{aligned}$$

We say that u is *Riemann integrable* (or (R)-*integrable*), if its lower integral coincides with its upper integral, and, in this case, we call *integral of* u (and write $\int_a^b u(t)\ dt$) their common value.

Definition 3.10 A measurable nonnegative map $f \in I\!R^X$ is *integrable* if there exists in R the quantity

(3.10.1) $\displaystyle\int_0^{+\infty} u(t)\ dt \equiv \sup_{a>0} \int_0^a u(t)\ dt = (o) - \lim_{a \to +\infty} \int_0^a u(t)\ dt,$

where the integral in (3.10.1) is intended as in Definition 3.9. If f is integrable, we indicate the element in (3.10.1) by the symbol $\displaystyle\int_X f\ dP$. If f is not necessarily positive, we say that a measurable function $f : X \to I\!R$ is *integrable* if there exist in R the following quantities:

$$I_1 \equiv \int_0^{+\infty} u(t)\ dt$$

and

$$I_2 \equiv (o) - \lim_{b \to -\infty} \int_b^0 [u(t) - P(X)]\ dt,$$

and in this case we set

$$\int_X f\ d\mu \equiv I_1 + I_2.$$

We indicate the quantity $\int_X f\ dP$ also by $E(f)$.

It is easy to check that this integral is well-defined, monotone, positively homogeneous, and satisfies the following properties, $\forall\ f \geq 0$ (see also [5], [8]):

a) $\int_X f\ dP = \int_X (f \wedge c)\ dP + \int_X f - (f \wedge c)\ dP,\ \forall\ c > 0.$

b) $\int_X f\ dP = \sup_{n \in I\!N} \int_X (f \wedge n)\ dP = (o) - \lim_{n \to \infty} \int_X (f \wedge n)\ dP.$

c) $\int_X f\ dP = (o) - \lim_{n \to \infty} \int_X (f \vee \frac{1}{n} - \frac{1}{n})\ dP.$

Conversely, let X be any set, and $B \subset [0, +\infty[^X$ such that $0 \in B$, and $f \wedge a,\ f \vee a - a \in B$, whenever $a \in [0, +\infty[$ and $f \in B$. If $T : B \to R$ is a monotone (positively homogeneous) "functional", satisfying $a), b)$ and $c)$, then there exists a monotone set function $P : \mathcal{P}(X) \to R$, such that $T(f) = \int_X f\ dP$, where the integral is intended as above.
See Representation Theorem in [16].

Remark 3.11 Let $P : \mathcal{A} \to R$ be a capacity, and Ω be as in 2.2. There exists a nowhere dense set $N \subset \Omega$, such that, $\forall\ \omega \notin N$, the map $P_\omega : \mathcal{A} \to \tilde{I\!R}$, defined by setting

$$P_\omega(A) \equiv P(A)(\omega),$$

is real-valued.
It is clear that, for each integrable function f, there exists a meager set M, depending only on f, such that, $\forall\ \omega \in \Omega \setminus M,\ \int_X\ f\ dP_\omega = (\int_X\ f\ dP)(\omega)$.

The following result holds:

Proposition 3.12 *Let $P : \mathcal{P}(X) \to R$ be a submodular capacity, and $f, g \in I\!R^X$ two nonnegative integrable maps. Then,*

$$\int_X\ (f+g)\ dP \leq \int_X\ f\ dP + \int_X\ g\ dP.$$

Proof: Let N and P_ω be as in Remark 3.11. It is clear that P_ω is a submodular capacity, $\forall\ \omega \in \Omega \setminus N$. By "Subadditivity Theorem" of [9], we have:

$$\int_X\ (f+g)\ dP_\omega \leq \int_X\ f\ dP_\omega + \int_X\ g\ dP_\omega,\ \forall\ \omega \notin N.$$

So, up to the complement of meager sets, one has:

$$[\int_X\ (f+g)\ dP](\omega) = \int_X\ (f+g)\ dP_\omega \leq$$

$$\leq \int_X\ f\ dP_\omega + \int_X\ g\ dP_\omega = [\int_X\ f\ dP](\omega) + [\int_X\ g\ dP](\omega).$$

Thus, the assertion follows. $\square$.
By using the same technique as above, it is easy to prove the following two propositions:

Proposition 3.13 *If $P : \mathcal{A} \to R$ is a mean, and f, g are integrable, then for every $A \in \mathcal{A}$*

$$\int_A\ (f+g)\ dP = \int_A\ f\ dP + \int_A\ g\ dP.$$

Proposition 3.14 *If $P : \mathcal{P}(X) \to R$ is a capacity, and f, g are integrable and comonotonic, then*

$$\int_X\ (f+g)\ dP = \int_X\ f\ dP + \int_X\ g\ dP.$$

The proof of the following results are analogous to [9]:

Proposition 3.15 *Let P_1 and P_2 be two capacities, and assume that $f \in I\!R^X$ is an integrable function, w. r. both to P_1 and P_2. Then,*

a') *For every $c \geq 0$, $\int_X\ f\ d(c\ P_1) = c\ \int_X\ f\ dP_1$.*

b') *$P_1 + P_2$ is a capacity, and $\int_X\ fd(P_1 + P_2) = \int_X\ fdP_1 + \int_X\ fdP_2$.*

c') *If $P_1(X) = P_2(X)$ or $f \geq 0$, then $[P_1 \leq P_2] \Longrightarrow [\int_X\ fdP_1 \leq \int_X\ fdP_2.]$*

Proof: the result follows from Theorem 2.2 and Proposition 5.2 of [9].

Proposition 3.16 *If $f \geq 0$ and $(P_n)_n$ is a sequence of capacities, such that $P_n \leq P_{n+1}\ \forall\ n$ and $(o) - \lim_n\ P_n(A) = P(A)\ \forall\ A \in \mathcal{A}$, then $(o) - lim_n\ \int_X\ f\ dP_n = \int_X\ f\ dP$.*

Proof: see 5.2.iv of [9].

Theorem 3.17 *Let $P : \mathcal{P}(X) \to R$ be a submodular capacity. Then, for every class $\mathcal{C}$ of integrable comonotonic functions, there exists a mean $\mu : \mathcal{P}(X) \to R$, such that*

$$P(A) \leq \mu(A), \ \forall \ A \in \mathcal{P}(X),$$

and

$$\int_X f \ d\mu = \int_X f \ dP, \ \forall \ f \in \mathcal{C}.$$

Proof: The proof is a direct consequence of the Hahn-Banach Theorem for Riesz-space-valued functionals (see Proposition 10.1. of [9] and [4]) .

4 The Riesz representation Theorem.

Throughout this section, R is a Dedekind complete Riesz space and X is any normal topological space, that is, such that every disjoint pair of closed sets can be separated by disjoint open sets. (We note that there exist some normal topological spaces which are not T_2 : see also [14]). From now on, we denote by $\mathcal{G} \equiv \{$ open subsets of $X\}$, and $\mathcal{F} \equiv \{$ closed subsets of $X\}$. We will prove a representation theorem for Riesz-space-valued functionals (For similar theorems existing in literature, see [10], [13], [22], [24]). We begin with a preliminary Lemma.

Lemma 4.1 *Let $\xi : \mathcal{G} \to R$ be a subadditive set function, and put $\mu(E) \equiv \inf\{\xi(V) : V \in \mathcal{G}, \ V \supset E\}$, $\forall \ E \in \mathcal{P}(X)$. Then, μ is subadditive.*

Proof: Let $E_1, \ E_2 \subset X$. Choose arbitrarily two open sets $V_i \supset E_i$ $(i = 1, 2)$. One has:

$$\mu(E_1 \cup E_2) \leq \xi(V_1 \cup V_2) \leq \xi(V_1) + \xi(V_2).$$

By arbitrariness of V_1 and V_2, we get

$$\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2),$$

that is, subadditivity of μ.

Lemma 4.2 *Let $\mu : \mathcal{P}(X) \to R$ be a subadditive capacity, such that $\mu(E) = \inf\{\mu(V) : V \supset E, V$ is open $\}$, $\forall \ E \in \mathcal{P}(X)$, and let $\mathcal{M} \equiv \{E \in \mathcal{P}(X) : \mu(E) = \sup\{\mu(K) : K \subset E, \ K$ is closed $\}\ \}$. Then $A \cap B^c \in \mathcal{M}$, whenever $A, \ B \in \mathcal{M}$.*

Proof: Let $A, B \in \mathcal{M}$.
There exist two nets of closed sets,

$$\{K^1_\alpha\}_\alpha, \{K^2_\alpha\}_\alpha, \ K^1_\alpha \subset A, \ K^2_\alpha \subset B \ \forall \ \alpha,$$

and two nets of open sets,

$$\{V^1_\beta\}_\beta, \{V^2_\beta\}_\beta, \ V^1_\beta \supset A, \ V^2_\beta \supset B \ \forall \ \beta,$$

such that

$$K^1_\alpha \subset A \subset V^1_\beta, \ K^2_\alpha \subset B \subset V^2_\beta, \ \forall \ \alpha, \ \beta,$$

and

$$\inf_{(\alpha,\beta)} \ \mu(V^i_\beta \cap (K^i_\alpha)^c) = (o) - \lim_{(\alpha,\beta)} \ \mu(V^i_\beta \cap (K^i_\alpha)^c) = 0.$$

As

$$A \cap B^c \subset V^1_\beta \cap (K^2_\alpha)^c \subset (V^1_\beta \cap (K^1_\alpha)^c) \cup (K^1_\alpha \cap (V^2_\beta)^c) \cup (V^2_\beta \cap (K^2_\alpha)^c) \ \forall \ \alpha, \ \beta,$$

then

$$\begin{aligned} 0 &\leq \mu(A \cap B^c) \leq (o) - \lim_{(\alpha,\beta)} \mu(V^1_\beta \cap (K^1_\alpha)^c) + (o) - \lim_{(\alpha,\beta)} \mu(K^1_\alpha \cap (V^2_\beta)^c) + (o) - \lim_{(\alpha,\beta)} \mu(V^2_\beta \cap (K^2_\alpha)^c) = \\ &= (o) - \lim_{(\alpha,\beta)} \mu(K^1_\alpha \cap (V^2_\beta)^c) = \sup_{(\alpha,\beta)} \mu(K^1_\alpha \cap (V^2_\beta)^c) \leq \sup \ \{\mu(H) : H \subset A \cap B^c, \ H \text{ closed } \}. \end{aligned}$$

On the other hand, it is easy to check that

$$\mu(A \cap B^c) \geq \sup \ \{\mu(H) : H \subset A \cap B^c, \ H \text{ closed } \}.$$

Therefore, $A \cap B^c \in \mathcal{M}$. $\square$

Theorem 4.3 *Let R be a Dedekind complete Riesz space, and assume that T is a positive linear R-valued mapping, defined on $\mathcal{C}_b(X)$. Then, there exists an algebra $\mathcal{M} \subset \mathcal{P}(X)$, containing all closed sets in X, and there exists a unique subadditive capacity $\mu : \mathcal{P}(X) \to R$, such that μ is a mean on $\mathcal{M}$, and:*

1. $T(f) = \int_X \ f \ d\mu, \ \forall \ f \in \mathcal{C}_b(X)$
2. $\mu(E) = \inf \ \{\mu(V) : E \subset V, \ V \textit{ open } \}, \ \forall E \in \mathcal{P}(X)$
3. $\mu(E) = \sup \ \{\mu(K) : E \supset K, \ K \textit{ closed } \}, \ \forall E \in \mathcal{M}$
4. *If $E \in \mathcal{M}, \ A \subset E$ and $\mu(E) = 0$, then $A \in \mathcal{M}$.*

Proof: We divide the proof in steps.

Step 1 Definition and subadditivity of μ.

For every open set V, set

$$\xi(V) \equiv \sup\{T(f) : f \prec V\}.$$

We note that this definition makes sense: indeed, for each open set V and for every $f \prec V$, one has: $T(f) \leq T(1)$, and so $\xi(V) \leq T(1)$, by virtue of monotonicity of T. For an arbitrary set $E \subset X$, put

$$\mu(E) \equiv \inf\{\xi(V) : E \subset V, \ V \in \mathcal{G}\}.$$

First, we remark that for every open subset $V \subset X, \ \mu(V) = \xi(V)$ and that μ is monotonic.
We now prove that

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$$

for each pair (V_1, V_2) of open sets. Fix arbitrarily $g \prec V_1 \cup V_2$. By a classical result, there exist two continuous real-valued functions $h_1, \ h_2$, such that $h_i \prec V_i (i = 1, 2)$ and $h_1(x) + h_2(x) = 1, \ \forall \ x \in$ *supp* g. So, $h_i g \prec V_i, \ g = h_1 g + h_2 g$, and hence

$$T(g) = T(h_1 g) + T(h_2 g) \leq \mu(V_1) + \mu(V_2).$$

By arbitrariness of g, we obtain subadditivity of μ on $\mathcal{G}$. Thus, by Lemma 4.1, μ is subadditive on $\mathcal{P}(X)$.

Step 2 Additivity of μ.

Set $\mathcal{M} \equiv \{E \subset X : \mu(E) = \sup \ \{\mu(K) : K \subset E : K \text{ is closed } \}$. We prove that $V \in \mathcal{M}$, for all open set V. In order to do this, it is enough to show that, for every $f \prec V$, there exists a closed set $K \subset V$, such that $\mu(K) \geq T(f)$.
Fix $f \prec V$ and denote by K the support of f. Moreover, let W be any open set, containing K. We have: $f \prec W$, and so $T(f) \leq \mu(W)$. Hence, $T(f) \leq \mu(K)$. As $K \subset V$, then $V \in \mathcal{M}$.
Now, we prove additivity of μ on $\mathcal{M}$. First of all, we prove that

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2),$$

whenever K_1 and K_2 are two closed disjoint subsets of X. By normality of X, there exist two open disjoint sets V_i, such that $V_i \supset K_i$ $(i = 1, 2)$. Fix arbitrarily an open set $W \supset K_1 \cup K_2$. By Theorem 2.2, there exists a meager set $N \subset \Omega$, such that

$$\mu(W \cap V_i)(\omega) = \sup \ \{T(f)(\omega) : f \prec W \cap V_i\}, \ \forall \ \omega \notin N \ (i = 1, 2).$$

So, for every $\varepsilon > 0$ and $\omega \notin N$, there exist some continuous functions $f_{1,\omega}, \ f_{2,\omega}$, such that $f_{i,\omega} \prec W \cap V_i$, and

$$T(f_{i,\omega})(\omega) > \mu(W \cap V_i)(\omega) - \frac{\varepsilon}{2}, \ i = 1, 2.$$

We have, up to the complement of meager sets,

$$\mu(K_1)(\omega) + \mu(K_2)(\omega) \leq \mu(W \cap V_1)(\omega) + \mu(W \cap V_2)(\omega) \leq$$

$$\leq T(f_1)(\omega) + T(f_2)(\omega) + \varepsilon \leq \mu(W)(\omega) + \varepsilon.$$

As the complement of a meager set is dense, we get

$$\mu(K_1)(\omega) + \mu(K_2)(\omega) \leq \mu(W)(\omega) + \varepsilon, \ \forall \ \omega \in \Omega.$$

By arbitrariness of ε, we obtain

$$\mu(K_1) + \mu(K_2) \leq \mu(W).$$

By arbitrariness of W, we deduce:

$$\mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2),$$

and therefore equality by the first step.

Now we prove that, if E_1 and E_2 are two disjoint sets, such that

$$\mu(E_i) = \sup \ \{\mu(K) : K \subset E_i, K \text{ is closed } \}, \ (i = 1, 2)$$

then $\mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2) = \sup\{\mu(K) : K \subset E_1 \cup E_2, \ K \text{ is closed } \}$.
To prove this, choose arbitrarily two closed sets $K_i \subset E_i$ $(i = 1, 2)$. We have:

$$\mu(E_1 \cup E_2) \geq \sup \ \{\mu(K) : K \subset E_1 \cup E_2\} \geq \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$$

and hence

$$\mu(E_1 \cup E_2) \geq \sup \ \{\mu(K) : K \subset E_1 \cup E_2\} \geq \mu(E_1) + \mu(E_2) \geq \mu(E_1 \cup E_2),$$

by virtue of subadditivity of μ. So, the inequalities above are the required equalities.
We now prove that $\mathcal{M}$ is an algebra. First of all, thanks to Lemma 4.2, $\mathcal{M}$ is closed with respect to set differences. Moreover, as $A \cup B = (A \cap B^c) \cup B$, by virtue of the previous result, it follows that $A \cup B \in \mathcal{M}$. Furthermore, as $A \cap B = A \cap (A \cap B^c)^c$, we have that $A \cap B \in \mathcal{M}$. So, $\mathcal{M}$ is an algebra.

Step 3 $T(f) = \int_X \; f \; d\mu, \; \forall \; f \in \mathcal{C}_b(X)$.

First of all, we observe that it will be enough to prove the inequality

(+) $\qquad T(f) \leq \int_X \; f \; d\mu, \; \forall \; f \in \mathcal{C}_b(X)$.

Indeed, by changing f with $-f$, from (+) we get:

$$T(f) = -T(-f) \geq - \int_X \; (-f) \; d\mu = \int_X \; f \; d\mu.$$

Let 's prove (+).
Fix $f \in \mathcal{C}_b(X)$, and let $[a, b]$ be an interval, containing the range of f. Choose arbitrarily $\varepsilon > 0$. Then, there exists a division $y_0 \equiv a < y_1 < \ldots < y_n \equiv b$, such that $y_i - y_{i-1} < \varepsilon, \; \forall \; i = 1, \ldots, n$. Set

$$E_i \equiv \{x \in X : y_{i-1} < f(x) \leq y_i\} \; (i = 1, \ldots, n).$$

As f is continuous, then $\{E_i\}_{i=1}^n$ is a partition of elements of $\mathcal{M}$. There exists a meager set $L \subset \Omega$, such that, for all $\omega \notin L$ and $i = 1, \ldots, n$, there exists open sets $V_i^\omega, \; V_i^\omega \supset E_i$, such that $\mu(V_i^\omega)(\omega) \leq \mu(E_i)(\omega) + \frac{\varepsilon}{n}$, and $f(x) < y_i + \varepsilon, \; \forall \; x \in V_i^\omega$. For each fixed $\omega \notin L$, let $(h_i^\omega)_i$ be a partition of the unity for $\{V_i^\omega\}_{i=1}^n$: we have that $f \equiv \sum_{i=1}^n \; h_i^\omega f$.
As $h_i^\omega \; f \leq (y_i + \varepsilon) \; h_i^\omega$, and $y_i + \varepsilon < f(x) + 2 \; \varepsilon$ on E_i, then one has:

$$\begin{aligned}
[T(f)](\omega) \quad &= \quad [\sum_{i=1}^n \; T(h_i^\omega \; f)](\omega) \leq [\sum_{i=1}^n \; (y_i + \varepsilon) T(h_i^\omega)](\omega) \; \leq \\
&\leq \quad [\sum_{i=1}^n \; (y_i + \varepsilon)\mu(V_i^\omega)](\omega) \leq \sum_{i=1}^n \; (y_i + \varepsilon)(\mu(E_i)(\omega) + \frac{\varepsilon}{n}) \leq \\
&\leq \quad \sum_{i=1}^n \; [\int_{E_i} \; f \; d\mu](\omega) + 2 \; \varepsilon \; [\mu(X)](\omega) + \varepsilon = [\int_X \; f \; d\mu](\omega) + \varepsilon(2 \; \mu(X)(\omega) + 1).
\end{aligned}$$

There exists a closed nowhere dense set $L^{'} \subset \Omega$, such that, $\forall \; \omega \notin L^{'}, \; [\mu(X)](\omega) \in R$. By arbitrariness of ε, we get

(++) $\qquad [T(f)](\omega) \leq [\int_X \; f \; d\mu](\omega), \; \forall \; \omega \notin L^{'}$.

As $\Omega \setminus L^{'}$ is (open and) dense, then (++) holds for all $\omega \in \Omega$. Thus, $T(f) \leq \int_X \; f \; d\mu, \; \forall \; f \in \mathcal{C}(\Omega)$.

Step 4 Uniqueness of μ.

Let μ_1 and μ_2 be two means, for which the assertion of Theorem 4.3 holds, and fix a closed set K. Choose arbitrarily an open set V : then, by Urysohn 's Lemma, there exists a continuous function f such that $K \prec f \prec V$. We have:

$$\mu_1(K) = \int_X \; \chi_K \; d\mu_1 \leq \int_X \; f \; d\mu_1 = T(f) =$$

$$= \int_X \; f \; d\mu_2 \leq \int_X \; \chi_V \; d\mu_2 = \mu_2(V).$$

By arbitrariness of V, we get:

$$\mu_1(K) \leq \inf\{\mu_2(V) : K \subset V, \; V \text{ open } \} = \mu_2(K).$$

Similarly, we can prove the opposite inequality, and so μ_1 and μ_2 coincide on the class of all closed sets, and, by construction, they are equal on the whole of $\mathcal{P}(X)$.
The proof of 4. is straightforward. So, the theorem is completely proved. $\square$

Remark 4.4 Under the same hypotheses as in Theorem 4.3, we can claim that there exists a mean $\nu : \mathcal{P}(X) \to R$, satisfying 1.), 3.), 4.) and 2.) of 4.3 for every $E \in \mathcal{M}$, where $\mathcal{M}$ is as in the proof of 4.3. Indeed, by well-known extension theorems, $\mu|_{\mathcal{M}}$ has a finitely additive extension $\nu : \mathcal{P}(X) \to R$. We can prove that $T(f) = \int_X f \, d\nu$, $\forall\, f \in \mathcal{C}_b(X)$: in fact, just sets in $\mathcal{M}$ are involved, so $\int_X f \, d\nu = \int_X f \, d\mu$ when $f \in \mathcal{C}_b(X)$.

Remark 4.5 A consequence of Theorem 4.3 and Proposition 3.6 is that, if X is a compact normal topological space, then μ and ν are σ-additive on $\mathcal{M}$. However, in general, we cannot obtain σ-additivity of μ or ν on the Borel σ-field (see also [24]).

5 Convergence in distribution.

Throughout this section, we will follow an approach similar to the ones in [3] and [14].

Let X be a normal topological space, $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{B}$ be the class of all Borel sets of X, and R be a Dedekind complete Riesz space.

We begin with the following:

Definition 5.1 Let $P_n,\ P : \mathcal{A} \to R$ be means. We say that P_n converges weakly to P, and write $P_n \Rightarrow P$, if

$$(o) - \lim_{n\to+\infty} \int_X f \, dP_n = \int_X f \, dP, \ \forall\, f \in \mathcal{C}_b(X).$$

We now prove the following characterization of weak convergence (see also [3], [14], [20]), which is a version of Portmanteau 's theorem.

Theorem 5.2 *If P is tight, then, the following conditions are equivalent:*

5.2.1.) $P_n \Rightarrow P$

5.2.2.) $(o) - \limsup_n \ P_n(F) \le P(F)$, *for all closed set F*

5.2.3.) $(o) - \liminf_n \ P_n(B) \ge P(B)$, *for all open set B*

Proof: The proof is analogous to the classical one.

A consequence of Theorem 4.3 is the following:

Theorem 5.3 *Let $(P_n : \mathcal{B} \to R)_n$ be a sequence of means, such that*

$$(o) - \lim_n \int_X f \, dP_n \in R, \ \forall\, f \in \mathcal{C}_b(X).$$

Then there exists a regular mean P, such that

$$(o) - \lim_n \int_X f \, dP_n = \int_X f \, dP, \ \forall\, f \in \mathcal{C}_b(X).$$

Proof: Let $T(f) \equiv (o) - \lim_n \int_X f\, dP_n, \ \forall\, f \in \mathcal{C}_b(X)$. It is easy to check that T satisfies the hypotheses of Theorem 4.3. So, there exists a regular set function μ, such that $T(f) = \int_X f\, d\mu$, $\forall\, f \in \mathcal{C}_b(X)$. Then, there exists a mean $P : \mathcal{P}(X) \to R$, regular on $\mathcal{M}$, coinciding with μ on the algebra $\mathcal{M}$ generated by open sets.
By proceeding as in Step 3 of Theorem 4.3, one readily shows that $\int_X f\, dP = \int_X f\, d\mu, \ \forall\, f \in \mathcal{C}_b(X)$. Thus, the assertion follows. □.

Remark 5.4 We notice that, if R is super Dedekind complete (that is R is Dedekind complete and every supremum of elements of R can be viewed as a suitable countable supremum), then every regular set function is tight.

Definition 5.5 We say that a capacity $P : \mathcal{A} \to R$ satisfies the *countable chain condition* (shortly, CCC) when for every family of pairwise disjoint sets $\mathcal{D}$, $\mathcal{D} \subset \mathcal{A}$, such that $P(D) \neq 0\ \forall\, D \in \mathcal{D}$, then $\mathcal{D}$ is countable (see [18]).

Observe that, if R is super Dedekind complete, then every measure P satisfies (CCC) (see [18]). Moreover, if P is a mean, satisfying (CCC), then, for each $f \in \mathbb{R}^X$, the set $\mathcal{V}_f \equiv \{\alpha \in \mathbb{R} : P(\{x \in X : f(x) = \alpha\}) \neq 0\}$ is countable.

Theorem 5.6 *Let* $(P_n)_n$ *be a sequence of means, and suppose that* P *is a tight mean, satisfying* (CCC). *Then, the following conditions are equivalent:*

5.6.1.) $P_n \Rightarrow P$

5.6.2.) $(o) - \lim_n P_n(A) = P(A)$, *for all subsets* $A \subset X$ *such that* $P(\partial A) = 0$.

Proof:
5.6.1.) $\Longrightarrow$ 5.6.2.) Applying 5.2.2.) and 5.2.3.), we have:

$$P(\overline{A}) \geq (o) - \limsup_n P_n(\overline{A}) \geq (o) - \limsup_n P_n(A) \geq (o) - \liminf_n P_n(A) \geq$$

$$\geq (o) - \liminf_n P_n(A^o) \geq P(A^o) = P(A) = P(\overline{A}),$$

if $P(\partial A) = 0$: thus, $P(A) = (o) - \lim_n P_n(A)$.
5.6.2.) $\Longrightarrow$ 5.6.1.) Let $f \in \mathcal{C}_b(X)$, and pick $\alpha, \beta \in \mathbb{R}$, such that $\alpha < f(x) < \beta$, $\forall\, x \in X$. Then, $\forall\, \varepsilon > 0$, we can find $\alpha \equiv \alpha_0 < \alpha_1 < \cdots < \alpha_k \equiv \beta$, $\alpha_i - \alpha_{i-1} < \varepsilon$, such that $P(\{x \in X : f(x) = \alpha_i\}) = 0$. Set

$$C_i \equiv \{x \in X : \alpha_{i-1} < f(x) \leq \alpha_i\}.$$

If y is a boundary point of C_i, then $f(y)$ is either α_{i-1} or α_i; hence, $P(\partial C_i) = 0$. One has:

$$\sum_{i=1}^{k} \alpha_{i-1}\, P_n(C_i) \leq \int_S f\, dP_n \leq \sum_{i=1}^{k} \alpha_i\, P_n(C_i),\ \forall\, n \in \mathbb{N};$$

$$\sum_{i=1}^{k} \alpha_{i-1}\, P(C_i) \leq \int_X f\, dP \leq \sum_{i=1}^{k} \alpha_i\, P(C_i),$$

As $P(\partial C_i) = 0$, then, by hypothesis, we have:

$$(o) - \lim_n \sum_{i=1}^{k} \alpha_i\, P_n(C_i) = \sum_{i=1}^{k} \alpha_i\, P(C_i);$$

$$(o) - \lim_n \sum_{i=1}^{k} \alpha_{i-1} \; P_n(C_i) = \sum_{i=1}^{k} \alpha_{i-1} \; P(C_i).$$

But

$$0 \leq \sum_{i=1}^{k} \alpha_i \; P_n(C_i) - \sum_{i=1}^{k} \alpha_{i-1} \; P(C_i) \leq \varepsilon \; P(X).$$

From this, it follows that

$$0 \leq | \int_X f \; dP - (o) - \limsup_n \int_X f \; dP_n | \leq \varepsilon \; P(X),$$

$$0 \leq | \int_X f \; dP - (o) - \liminf_n \int_X f \; dP_n | \leq \varepsilon \; P(X).$$

So, $(o) - \lim_n \int_X f \; dP_n = \int_X f \; dP$, and the theorem is completely proved. □

Let X, Y and P be as in 5.2, and $\mathcal{B}$ $[\mathcal{E}]$ be the class of all Borelian subsets of X $[Y]$. Given a measurable map $h : X \to Y$, and a mean $P : \mathcal{B} \to R$, define $Ph^{-1} : \mathcal{E} \to R$, by setting $Ph^{-1}(A) \equiv P(h^{-1}A), \; \forall \; A \in \mathcal{E}$. It is easy to prove the following:

Theorem 5.7 *Let $(P_n)_n$ be a sequence of means, and suppose that P is a tight mean. Let $h : X \to Y$ be a measurable mapping, and denote by D_h the set of discontinuities of h. If $P_n \Rightarrow P$ and $P(D_h) = 0$, then $P_n h^{-1} \Rightarrow Ph^{-1}$.*

The following result holds.

Theorem 5.8 *Let $P : \mathcal{B} \to R$ be a mean, and assume that $f : I\!R \to I\!R$ is a measurable bounded map, and $h : X \to I\!R$ is a measurable function. Then,*

5.8.1.) $\int_X f \circ h \; dP = \int_{I\!R} f \; dPh^{-1}$,

provided that both of members make sense.

Proof: Straightforward.

We now state the following (see also [3]):

Theorem 5.9 *Let $P_n, P : \mathcal{A} \to R$ be means, and suppose that P is tight. Assume that $P_n \Rightarrow P$ and $h : X \to I\!R$ is a bounded measurable function, such that $P(D_h) = 0$.*
Then, $(o) - \lim_{n \to +\infty} \int_X h \; dP_n = \int_X h \; dP$.

Let now P be a measure. Given a random variable f , we call *distribution of* f associated with P the set function P_f, defined by setting $P_f(A) \equiv P(f^{-1}(A))$, for all Borel sets $A \subset I\!R$.

Definition 5.10 We say that the sequence $(f_n)_n$ *converges in distribution* to f if $P_{f_n} \Rightarrow P_f$.

We begin with a Fatou's-type theorem.

Theorem 5.11 *Let $(f_n)_n$ be a sequence of random variables, convergent in distribution to a random variable f, and assume that P is a tight mean, satisfying* (CCC). *Then,*

$$\int_X |f| \; dP \leq (o) - \liminf_n \int_X |f_n| \; dP.$$

Proof: (see also [3]) For every $\alpha \in I\!R^+$, choose

$$h_\alpha(x) \equiv \begin{cases} |x|, & \text{if } |x| \leq \alpha \\ \\ 0, & \text{if } |x| > \alpha \end{cases}$$

If α is such that $P(\{x \in X : |f(x)| = \alpha\}) = 0$, then we get

$$\int_{\{x\in X:|f(x)|\leq\alpha\}} |f| \, dP = (o) - \lim_{n\to+\infty} \int_{\{x\in X:|f_n(x)|\leq\alpha\}} |f_n| \, dP \leq$$
$$\leq \quad (o) - \liminf_n \int_X |f_n| \, dP,$$

by virtue of Theorem 5.7. As

$$(o) - \lim_{\alpha\to+\infty} \int_{\{x\in X:|f_n(x)|\leq\alpha\}} |f| \, dP = \sup_{\alpha\in I\!R^+} \int_{\{x\in X:|f_n(x)|\leq\alpha\}} |f| \, dP = \int_X |f| \, dP \ ,$$

(see also [5]), then the assertion follows. □

Definition 5.12 We say that the sequence $(f_n)_n$ is *uniformly integrable* if

$$\sup_{n\in I\!N} \int_X |f_n| \, dP \in R,$$

and

$$(o) - \lim_{\alpha\to+\infty} [(o) - \limsup_n \int_{\{x\in X:|f_n(x)|\geq\alpha\}} |f_n| \, dP] = 0.$$

Now, we state a Vitali-type theorem for the involved integral, with respect to convergence in distribution (see also [3]).

Theorem 5.13 *Let $(f_n)_n$ be a uniformly integrable sequence of random variables, convergent in distribution to a random variable f. Assume that P is a measure, satisfying* (CCC).
Then,

$$(o) - \lim_{n\to+\infty} \int_X f_n \, dP = \int_X f \, dP. \tag{1}$$

Moreover, if $0 \leq f_n$, f are integrable, $(f_n)_n$ converges in distribution to f, and (1) holds, then $(f_n)_n$ is uniformly integrable.

Proof: (see also [3]) By hypothesis, we have:

$$\sup_n \int_X |f_n| \, dP \in R.$$

So, by virtue of Theorem 5.11, it follows that f is integrable. Set now, for every $\alpha > 0$,

$$h_\alpha(x) \equiv \begin{cases} x, & \text{if } |x| < \alpha, \\ \\ 0, & \text{if } |x| \geq \alpha \end{cases} \qquad Z_\alpha(x) \equiv \quad \{x \in X : f(x) = \alpha\}$$

By convergence in distribution of $(f_n)_n$ to f and Theorem 5.9, if $P(Z_\alpha) = 0$, we get:

$$(o) - \lim_{n\to+\infty} \int_X h_\alpha \circ f_n \, dP = \int_X h_\alpha \circ f \, dP.$$

Moreover,

$$\begin{aligned} \int_X f_n \, dP &= \int_X h_\alpha \circ f_n \, dP + \int_{\{x \in X: |f_n(x)| \geq \alpha\}} f_n \, dP; \\ \int_X f \, dP &= \int_X h_\alpha \circ f \, dP + \int_{\{x \in X: |f(x)| \geq \alpha\}} f \, dP. \end{aligned}$$

Let $W \equiv \{\alpha \in I\!\!R^+ : P(Z_\alpha) = 0\}$. Then,

$$\begin{aligned} (o) - \limsup_n \, |\int_X f \, dP - \int_X f_n \, dP| &= (o) - \lim_{\alpha \to +\infty, \ \alpha \in W} (o) - \limsup_n \, |\int_X f \, dP - \int_X f_n \, dP| \leq \\ &\leq (o) - \lim_{\alpha \to +\infty, \ \alpha \in W} (o) - \limsup_{n \in I\!\!N} \int_{\{x \in X: |f_n(x)| \geq \alpha\}} |f_n| \, dP + \\ &+ (o) - \lim_{\alpha \to +\infty, \ \alpha \in W} \int_{\{x \in X: |f(x)| \geq \alpha\}} |f| \, dP. \end{aligned}$$

From uniform integrability of $(f_n)_n$ and fundamental properties of the (monotone) integral, (1) follows.

Conversely, if $f_n, f \geq 0$ are integrable and satisfy (1), then, by virtue of the previous step, we get:

$$(o) - \lim_{n \to +\infty} \int_{\{x \in X: |f_n(x)| \geq \alpha\}} f_n \, dP = \int_{\{x \in X: |f(x)| \geq \alpha\}} f \, dP, \ \forall \ \alpha \in W.$$

So,

$$\begin{aligned} 0 &\leq (o) - \limsup_n \int_{\{x \in X: f_n(x) \geq \alpha\}} f_n \, dP \leq (o) - \lim_n \int_{\{x \in X: f_n(x) \geq \alpha\}} f_n \, dP - \int_{\{x \in X: f(x) \geq \alpha\}} f \, dP + \\ &+ \int_{\{x \in X: f(x) \geq \alpha\}} f \, dP = \int_{\{x \in X: f(x) \geq \alpha\}} f \, dP, \ \forall \ \alpha \in W. \end{aligned}$$

Hence,

$$(o) - \lim_{\alpha \in W} (o) - \limsup_n \int_{\{x \in X: |f_n(x)| \geq \alpha\}} f_n \, dP = 0.$$

As the net

$$\{(o) - \limsup_n \int_{\{x \in X: |f_n(x)| \geq \alpha\}} f_n \, dP\}_{\alpha \in I\!\!R^+}$$

is decreasing as α increases, then

$$(o) - \lim_{\alpha \to +\infty} (o) - \limsup_n \int_{\{x \in X: |f_n(x)| \geq \alpha\}} f_n \, dP = 0. \ \Box$$

A consequence of Theorem 5.13 is the following:

Theorem 5.14 (Dominated convergence Lebesgue theorem) *Let $(f_n)_n$ be a sequence of random variables, convergent in distribution to a random variable f. Assume also* (CCC). *Moreover, suppose that there exists an integrable random variable h such that $|f_n(x)| \leq |h(x)|$, for P-almost all $x \in X$.*
Then,

$$(o) - \lim_{n \to +\infty} \int_X f_n \, dP = \int_X f \, dP.$$

Remark 5.15 We note that the hypotheses of Theorem 5.13 are not enough to get convergence in L^1, even if $R = I\!\!R$ and the functions f_n and f are nonnegative; therefore, in general, Scheffé 's theorem does not hold. Indeed, any sequence of (uniformly bounded) random variables, convergent in distribution but not in probability, will give an example.

We now investigate some relations between convergence in distribution and convergence in measure.

Definition 5.16 Let R be a Dedekind complete Riesz space. Given a random variable f , we call *distribution function of* f the function $F_f : I\!R \to R$, defined by setting $F_f(x) \equiv P(\{z \in X : f(z) \leq x\}),\ x \in I\!R$.

Similarly as in the real case, it is easy to prove the following

Proposition 5.17 *If P is σ-additive, then the distribution function F_f satisfies the following properties:*

5.17.1.) *F_f is an increasing function.*

5.17.2.) $(o) - \lim_{x \to -\infty} F_f(x) = 0;\ \ (o) - \lim_{x \to +\infty} F_f(x) = P(X)$

5.17.3.) *F_f is right-continuous at every point $x \in I\!R$.*

Proposition 5.18 *Let $(f_n)_n$ be a sequence of random variables, convergent in measure to f. Then,*

5.18.1) $(o) - \lim_n F_{f_n}(x) = F_f(x)$, *for every $x \in I\!R$, such that F_f is continuous at x.*

Conversely, if $f(x) \equiv a\ \forall\ x \in X$, and condition 5.18.1) *holds, then $(f_n)_n$ converges in measure to f.*

Proof: The proof is analogous to the classical one.

Theorem 5.19 *For every $n \in I\!N_0$, let $P_n : \mathcal{B} \to R$ be a mean, such that the sequence $(P_n(I\!R))_n$ is bounded. Set $\Phi_n(x) \equiv P_n(]-\infty, x])$, and assume that $P_0^*(A) \equiv \inf\{P_0(V) : V$ open, $V \supset A\}$ satisfies* (CCC). *Moreover, suppose that*

5.19.1.)

$$(o) - \lim_{x \to -\infty} P_0(] - \infty, x]) = (o) - \lim_{x \to +\infty} P_0([x, +\infty[) = 0.$$

Then, the following are equivalent:

5.19.2.) *$(P_n)_n$ weakly converges to P_0.*

5.19.3.) $(o) - \lim_n \Phi_n(x) = \Phi_0(x)$, *for each continuity point x of Φ_0.*

Proof: The proof is straightforward.

We note that, in general, condition 5.19.1.) is strictly weaker than σ-additivity of P_0 (see [15]), but it cannot be dropped, even if $R = I\!R$ (see also [15]), in order to prove the implication [5.19.3.) $\Longrightarrow$ 5.19.2.)].

6 Laws of large numbers.

From now on, we assume that P is a measure. We begin with the following definition:

Definition 6.1 Let R be a Dedekind complete Riesz space, and $(a_n)_n$ a sequence in R. We call *series associated with* (a_n) the sequence (S_n), defined by setting

$$\begin{cases} S_1 = a_1 \\ \\ S_n = S_{n-1} + a_n, \quad n \in \mathbf{N}, \end{cases}$$

and we indicate this series by the symbol $\sum_{n=1}^{\infty} a_n$. We say that the series $\sum_{n=1}^{\infty} a_n$ *converges to $L \in R$ if* $L = (o) - \lim_n S_n$.

We introduce the following condition:

H1) For every $i \in I\!N$, for each A_i belonging to the σ-algebra $\sigma(f_1, \ldots, f_i)$ generated by $f_1, \ldots, f_i$, $\forall\ j > i \geq h \in I\!N$, it holds:

$$\int_{A_i} f_h\ f_j\ dP = 0.$$

We observe that, in the real case, H1) is equivalent to the following hypothesis:

$$E(f_{n+j}|\sigma(f_1, \ldots, f_{n-1})) = 0,\ \forall\ n,\ j \in I\!N.$$

Remark 6.2 We note that it is not advisable to proceed in terms of conditional expectation.
Indeed, let $R \equiv I\!R^2$, $P : \mathcal{A} \to R$ be a measure, $P = (P_1, P_2)$, where $P_1(X) = P_2(X) = 1$, and assume that $\int_X\ f\ dP_1 \neq \int_X\ f\ dP_2$.
Put $\mathcal{B} \equiv \{\emptyset, X\}$: then, it is easy to check that a mapping $g \in I\!R^X$ is $\mathcal{B}$-measurable if and only if it is constant. Then,

$$c = c\ P_j(X) = \int_X\ f\ dP_j\ (j = 1, 2)$$

and so

$$\int_X\ f\ dP_1 = \int_X\ f\ dP_2 :$$

contradiction.
Thus, in this case, we cannot define a real-valued map g, playing the same role, as the conditional expectation in the case $R = I\!R$.

The main result of this section is the strong law of large numbers.

Theorem 6.3 *Let $(f_n)_n$ be a sequence of random variables, such that f_n^2 is integrable, $\forall\ n$, and satisfying condition H1). Moreover, suppose that the series $\sum_{i=1}^{\infty} \frac{E(f_j^2)}{j^2}$ converges. Set $\bar{f}_n \equiv \frac{1}{n}\ \sum_{i=1}^{n}\ f_i$. Then, the sequence $(\bar{f}_n)_n$ converges to 0 almost everywhere.*

In order to prove this theorem, we introduce two Lemmas.

Lemma 6.4 *Let $f_1, \ldots, f_n, \ldots$ be random variables, satisfying H1), and suppose that f_n^2 is integrable, $\forall\ n \in I\!N$. If $S_j \equiv \sum_{i=1}^{j}\ f_i$, and $u_1 \leq \ldots \leq u_n$ are positive real numbers, then*

$$P(x \in X : |S_j(x)| < u_j,\ \forall\ j) \geq P(X) - \sum_{j=1}^{n}\ \frac{E(f_j^2)}{u_j^2}.$$

Proof. First of all, we prove that

(*) $\qquad E((f_h + f_j)^2) = E(f_h^2) + E(f_j^2), \forall\ h, j \in I\!N,\ h \neq j.$

Without loss of generality, we can suppose that $h < j$. In H1), choose $i = h$, and $A_i = X$; then, $E(f_h\ f_j) = 0$, and so (*) follows.
From (*) we obtain:

$$E(S_n^2) = \sum_{i=1}^{n}\ E(f_i^2).$$

Set now $\alpha_i \equiv \frac{1}{u_i^2}$, $\forall\ i = 1, \ldots, n;\ \alpha_{n+1} \equiv 0$, and

$$T(x) \equiv \sum_{j=1}^{n}\ (\alpha_j - \alpha_{j+1})\ [S_j(x)]^2,\ \forall\ x \in X.$$

It is:

$$E(T) = \sum_{j=1}^{n}\ (\alpha_j - \alpha_{j+1})\ E(S_j^2).$$

For every j, put

$$B_j \equiv \{x \in X : |S_i(x)| < u_i,\ \forall\ i < j; |S_j(x)| \geq u_j\},$$

and set

$$A \equiv \{x \in X : |S_j(x)| < u_j,\ \forall\ j = 1, \ldots, n\}.$$

It is easy to check that $B_i \cap B_l = \emptyset$, $\forall\ i \neq l$, and $\cup_{j=1}^{n}\ B_j = A^c$.
For every i and j, with $i < j$, we have:

$$\int_{B_i} S_j^2\ dP \geq \int_{B_i} S_i^2\ dP$$

by virtue of H1), and hence

$$E(S_j^2) \geq \sum_{i=1}^{j}\ u_i^2\ P(B_i).$$

Thus,

$$\sum_{j=1}^{n}\ \frac{1}{u_j^2}\ E(f_j^2) = \sum_{j=1}^{n}\ (\alpha_j - \alpha_{j+1})\ E(S_j^2) \geq \sum_{j=1}^{n}\ (\alpha_j - \alpha_{j+1})\ \sum_{i=1}^{j}\ u_i^2\ P(B_i) =$$

$$= \sum_{i=1}^{n}\ \alpha_i\ u_i^2\ P(B_i) = \sum_{i=1}^{n}\ P(B_i) = P(X) - P(A),$$

that is the assertion.

Lemma 6.5 *Let R be a Dedekind complete Riesz space, $R \ni \alpha_1, \ldots, \alpha_n \geq 0$ be such that the series $\sum_{j=1}^{\infty}\ \frac{\alpha_j}{j^2}$ converges, and set $\sigma_n = \sum_{j=1}^{n}\ \alpha_j$. Then, $(o) - \lim_n\ \frac{\sigma_n}{n^2} = 0$.*

Proof. Fix $n \in I\!N$, and set $k \equiv [\sqrt{n}]$. It holds:

$$0 \leq \frac{\sigma_n}{n^2} = \frac{1}{n^2}\ \sum_{j=1}^{k}\ \alpha_j + \frac{1}{n^2}\ \sum_{j=k+1}^{n}\ \alpha_j \leq \frac{1}{n^2}\ \sum_{j=1}^{k}\ \frac{k^2}{j^2}\ \alpha_j +$$

$$+ \sum_{j=k+1}^{n}\ \frac{\alpha_j}{j^2} = \frac{k^2}{n^2}\ \sum_{j=1}^{k}\ \frac{\alpha_j}{j^2} + \sum_{j=k+1}^{n}\ \frac{\alpha_j}{j^2} \leq$$

$$\leq \frac{1}{n}\ \sum_{j=1}^{\infty}\ \frac{\alpha_j}{j^2} + \sum_{j=k}^{\infty}\ \frac{\alpha_j}{j^2} \longrightarrow 0.$$

Thus, the assertion follows.

We are now able to prove the strong law of large numbers (Theorem 6.3).

Proof: Fix $n, m \in I\!N$ and $\varepsilon > 0$, and set

$$S_n \equiv \sum_{j=1}^{n} f_j, \; Y_1 \equiv S_n, \; Y_2 \equiv f_{n+1}, \ldots, Y_m \equiv f_{n+m-1}.$$

It is easy to see that the maps Y_j $(j = 1, \ldots, m)$ satisfy the hypotheses of Lemma 6.4.

For every $j = 1, \ldots, m$ put $T_j \equiv \sum_{i=1}^{j} Y_i$. Set

$$C^{\varepsilon}_{n,m} \equiv \{x \in X : |\bar{f}_j(x)| < \varepsilon \; \forall \; j = n, n+1, \ldots, n+m\}.$$

Then we have

$$C_{n,m} = \{x \in X : |T_j(x)| < (j+n-1)\varepsilon \; \forall \; j = 1, \ldots, m\}.$$

Hence, by Lemma 6.4, we get:

$$P(C^{\varepsilon}_{n,m}) \geq P(X) - \sum_{j=1}^{m} \frac{E(Y_j^2)}{(j+n-1)^2 \, \varepsilon^2} = P(X) - \frac{E(S_n^2)}{n^2 \, \varepsilon^2} -$$

$$- \sum_{j=n+1}^{m-1} \frac{E(f_j^2)}{j^2 \, \varepsilon^2} \geq P(X) - \frac{E(S_n^2)}{n^2 \, \varepsilon^2} - \sum_{j=n+1}^{\infty} \frac{E(f_j^2)}{j^2 \, \varepsilon^2}.$$

So, there exists a sequence $(p_n)_n(\varepsilon) \in R, \; p_n \downarrow 0$, such that

$$P(C^{\varepsilon}_{n,m}) \geq P(X) - \frac{E(S_n^2)}{n^2 \, \varepsilon^2} - \frac{p_n}{\varepsilon^2},$$

and hence

$$P(\cap_m \, C^{\varepsilon}_{n,m}) = \inf_m \; P(C^{\varepsilon}_{n,m}) \geq P(X) - \frac{E(S_n^2)}{n^2 \, \varepsilon^2} - \frac{p_n}{\varepsilon^2}.$$

Let $E^{\varepsilon}_n \equiv \cap_{m=1}^{\infty} \, C^{\varepsilon}_{n,m}$: then

$$P(\cup_n \, E^{\varepsilon}_n) = (o) - \lim_n \; P(E^{\varepsilon}_n) \geq P(X) - (o) - \lim_n \; \frac{E(S_n^2)}{n^2 \, \varepsilon^2}.$$

By hypotheses (see also Lemma 6.4), we have:

$$E(S_n^2) = \sum_{j=1}^{n} E(f_j^2), \; \forall \; n \in I\!N,$$

and so

$$(o) - \lim_n \frac{E(S_n^2)}{n^2 \, \varepsilon^2} = 0.$$

Thus,

$$P(X) \geq P(\cup_n \; \cap_{j \geq n} \; \{x \in X : |\bar{f}_j(x)| < \varepsilon\}) \geq P(X),$$

and therefore the sequence $(\bar{f}_n)_n$ converges to 0 almost everywhere. □

Now we state a version of the weak law of large numbers, and we observe that the assertion still is true, even if we assume that P is only a finitely additive positive R-valued set function.

Theorem 6.6 *Let $(f_n)_n$ be a sequence of random variables, such that $E(f_n) = 0$ and f_n^2 is integrable, $\forall\ n \in \mathbb{N}$, and suppose that*

$$E((f_n + f_m)^2) = E(f_n^2) + E(f_m^2),\ \forall\ n, m \in \mathbb{N}.$$

If $(o) - \lim_n \frac{1}{n^2} \sum_{i=1}^n E(f_i^2) = 0$, then $(o) - \lim_n E(\bar{f_n}^2) = 0$, where $\bar{f_n}$ is as in Theorem 6.3.

Proof: We have:

$$E(\bar{f_n}^2) = \frac{1}{n^2}\ E((\sum_{i=1}^n\ f_i)^2) = \frac{1}{n^2} \sum_{i=1}^n\ E(f_i^2).$$

By virtue of the hypotheses, we get:

$$(o) - \lim_n\ E(\bar{f_n}^2) = 0,$$

that is the assertion. □

Remark 6.7 We consider now the case when $(X_n)_n$ is a sequence of random variables defined on a probability space (Ω, Σ, P) and $\mathcal{Z}$ is a sub-σ-algebra of Σ.
For every $A \in \Sigma$ we set $\widetilde{P}(A) = P(A|\mathcal{Z}) = E(1_A|\mathcal{Z})$. Then $\widetilde{P} : \Sigma \to L_0$. Moreover, if $X \in L_1$, we can define $\widetilde{E}(X) = E(X|\mathcal{Z}) \in L_1 \subset L_0$. $\widetilde{E}(X)$ is a random variable obtained by integrating X with respect to $\widetilde{P}$.
In fact, given X in L_1, we can consider a sequence of random variables $(X_n)_n$ such that $X_n(\omega) \leq X_{n+1}(\omega)$ for almost every ω and $\sup_n X_n = X$ a.e. Then, using Beppo Levi's Theorem,

$$E(X|\mathcal{Z}) = \sup_n E(X_n|\mathcal{Z}) = \lim_n E(X_n|\mathcal{Z}).$$

So $\widetilde{P} : \Sigma \to L_1(\Omega, \mathcal{Z}, P) \subset L_0(\Omega, \mathcal{Z}, P) \subset L_0(\Omega, \Sigma, P)$ is a measure (observe in fact that if $A_n, A \in \Sigma$ such that $A_n \uparrow A$ it is $1_{A_n} \uparrow 1_A$ and so $E(1_{A_n}|\mathcal{Z}) \uparrow E(1_A|\mathcal{Z})$.) Finally, if X in $L_1(\Omega, \Sigma, P)$, then $\int X d\widetilde{P} = E(X|\mathcal{Z})$. This is obvious for simple functions, and it is possible to obtain for the general case using classical techniques.
If we suppose that $\widetilde{E}(X_h X_j 1_{A_i}) = E(X_h X_j 1_{A_i}|\mathcal{Z}) \equiv 0$ for every $j > i \geq h, A_i \in \sigma(X_1, \cdots, X_i)$ (this hypothesis is stronger then H1)), in fact it means that $E(X_j X_h|\mathcal{Z} \vee \sigma(X_1, \cdots, X_i)) = X_h E(X_i|\mathcal{Z} \vee \sigma(X_1, \cdots, X_i)) = 0$; and if

$$\sum \frac{1}{j^2} E(X_j^2|\mathcal{Z}) = \sum \frac{\widetilde{E}(X_j^2)}{j^2} \in L_0$$

then using Theorem 6.3 $\overline{X_n}$ converges to 0 $\widetilde{P}$ almost everywhere, i.e. $P(A|\mathcal{Z}) = 0$ where $A = \{\omega : \overline{X} \to 0\}$. This implies that $P(A) = 0$. Conversely if $P(A) = 0$ then $P(A|\mathcal{Z}) = 0$. So

$$\overline{X}_n \to 0\ \widetilde{P} - a.e. \text{ if and only if } \overline{X}_n \to 0\ P - a.e..$$

So, we note that, in this particular case, the classical result given in Theorem 6.3, for $R = \mathbb{R}$, is obtained by modifying the hypothesis: in particular by strengthening H1) and by weakening the convergence of $\sum \frac{1}{j^2} E(X_j^2|\mathcal{Z})$

References

[1] C.D.ALIPRANTIS - O.BURKINSHAW "Locally solid Riesz spaces", (1978), Academic Press.

[2] S.J. BERNAU "Unique representation of Archimedean lattice groups and normal Archimedean lattice rings", Proc. London Math. Soc., **15** (1965), 599-631.

[3] P. BILLINGSLEY "Convergence of Probability Measures," John Wiley and Sons (1968)

[4] A. BOCCUTO - D. CANDELORO "Sandwich Theorems, extension principles and amenability", Atti Sem. Mat. Fis. Univ. Modena, **42** (1994), 257-271.

[5] A. BOCCUTO–A. R. SAMBUCINI "On the De Giorgi-Letta integral with respect to means with values in Riesz spaces" Real Analysis Exchange **21** (1) 1995/6, 1-21.

[6] J.K. BROOKS–A. MARTELLOTTI "On the De Giorgi-Letta integral in infinite dimensions", Atti Sem. Mat. Fis. Univ. Modena **4** (1992) 285-302.

[7] G. COLETTI–G. REGOLI "Probabilità qualitative non archimedee e realizzabilità, Rivista di Matematica per le Scienze economiche e sociali, Fasc. I e II (1983), 79-99."

[8] E. De GIORGI–G. LETTA "Une notion générale de convergence faible pour des fonctions croissantes d'ensemble", Ann. Scuola Sup. Pisa **33** (1977) 61-99.

[9] D. DENNEBERG "Non-additive measure and integral", Kluwer (1994)

[10] R. DYCKERHOFF–K. MOSLER "Stochastic dominance with nonadditive probabilities", ZOR Methods and Models of Operations Research **37** (1993) 231-256.

[11] P. C. FISHBURN "The axioms of Subjective Probability", Statistical Science **1** (1986), 335-345.

[12] P. C. FISHBURN "The axioms and algebra of ambiguity", Theory and Decision **34** (1993) 119-137.

[13] I. GILBOA–D. SCHMEIDLER "Additive representation of non-additive measures and the Choquet integral", Ann. Oper. Research, **52** (1994), 43-65.

[14] B. GIROTTO–S. HOLZER "Weak convergence of masses on normal topological spaces," Sankhya, Ser. A **55**, No.2, (1993) 188-201.

[15] B. GIROTTO–S. HOLZER "Convergence in distribution of masses on the real line," Rend. Mat., Serie VII **9** (1989), 53-65.

[16] G. H. GRECO "Sulla rappresentazione di funzionali mediante integrali", Rend. Sem. Mat. Univ. Padova, **66** (1982) 21-42.

[17] M.C. ISIDORI–A. MARTELLOTTI–A.R. SAMBUCINI "Monotone Integral" (1995), preprint.

[18] Z. LIPECKI, "On unique extensions of positive additive set functions", Arch. Math. **41** (1983), 71-79.

[19] X. LIU - G. ZHANG, "Lattice-valued fuzzy measure and lattice-valued fuzzy integral", Fuzzy Sets and Systems, **62** (1994), 319-332.

[20] M. MÄRZ–R. M. SHORTT, "Weak convergence of vector measures", Publ. Math. Debrecen **45** (1994), 71-92.

[21] W. RUDIN "Real and complex Analysis," McGraw-Hill (1966).

[22] M. SCARSINI "Dominance conditions in non-additive expected utility theory", J. of Math. Econ. **21** (1992), 173-184.

[23] D. SCHMEIDLER "Integral representation without additivity, Proc. Am. Math. Soc. **97** (1986), 255-261."

[24] J. D. M. WRIGHT "Stone-algebra-valued measures and integrals", Proc. Lond. Math. Soc., **19** (1969), 107-122.

[25] J. D. M. WRIGHT "The measure extension problem for vector lattices," Ann. Inst. Fourier **21** (1971), 65-85.