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Results in Mathematics

Integration by Parts for Perron Type Integrals of Order 1 and 2 in Riesz Spaces

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Abstract. A Perron-type integral of order 1 and 2 for Riesz-space-valued functions is investigated. Some versions of integration by parts formula for this integral are proved for both orders.

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1. Introduction

A Perron-type integral of order $k \in \mathbb{N}$, for the case of real-valued functions, was introduced by R. D. James [16,17], see also [24] in connection with some applications to the problem of recovering the coefficients of trigonometric series by generalized Fourier formulae. Since then this type of integrals was studied in numerous papers [1, 9–12]. U. Das and A. G. Das gave in [14] some versions of integration by parts formula for this integral. Among related tools, used in this theory, we mention some generalizations of the concept of convexity and divided differences [9, 20, 21].

Perron-type integrals for Riesz-space-valued functions were defined in [4] and [6]. In the present paper we investigate some fundamental properties of the Perron integrals of order 1 and 2 in the Riesz-space-valued case and obtain a version of integration by parts formulae for these integrals. For technical reasons, in our paper the involved major and minor functions of order 1 and 2 are taken to be regular enough. This gives us an opportunity to replace, in the definition of the second order integral on $[a, b] \subset \mathbb{R}$, the "boundary"-type conditions at the points a and b, by the "initial"-type conditions at the point a. In this respect our definition is similar to the one used by P. S. Bullen [10] in the real-valued case and is slightly different from the one adopted by R. D. James in [16]. In the real-valued case it is known that some of regularity assumptions we are imposing here on major and

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minor functions do not make the class of the integrable functions smaller (see for example [8,22]). In the case of general Riesz spaces even the problem whether the Perron integral of order 1, defined by continuous major and minor functions, is equivalent to the one defined without the continuity conditions, is still open.

An important tool used here is the Maeda–Ogasawara–Vulikh Theorem on representation of Archimedean Riesz spaces as suitable spaces of continuous functions (see [2,7,19]). Another important notion on which our definition of the major and minor functions is based, is the one of the global limit studied in Section 3. Related notions of the global derivative and the global continuity are defined in Section 4. Section 5 contains some information on Riemann and Riemann–Stieltjes integrals for the Riesz-space-valued case. The definitions of the Perron integrals of order 1 and 2 and some properties of these integrals are given in Sections 6 and 7. The main results of the paper, related to the integration by parts formulae are obtained in Section 8.

2. Preliminaries

Let R be a Riesz space. If $y \in R$, we say that $y > 0$ when $y \geq 0$ and $y \neq 0$. We denote by R^+ and R_0^+ the sets of those elements $y \in R$ such that $y > 0$ and $y \geq 0$ respectively. We add to R two extra elements, $\pm\infty$, extending in a natural way ordering and operations and denoting $\overline{R} = R \cup \{\pm\infty\}$. By convention, we will say that the supremum of any unbounded above nonempty subset of R is $+\infty$ and the infimum of any unbounded below nonempty subset of R is $-\infty$.

A Dedekind complete Riesz space R is said to be *super Dedekind complete* if every nonempty subset R_1 bounded from above contains a countable subset with the same supremum as R_1.

Given a net $(y_\eta)_{\eta\in\Lambda}$ in $\overline{R}$, where $(\Lambda, \geq) \neq \emptyset$ is a directed set, let

$$\limsup_{\eta} y_\eta = \inf_{\eta}, [\sup_{\zeta\geq\eta} y_\zeta]\,, \quad \liminf_{\eta} y_\eta = \sup_{\eta} [\inf_{\zeta\geq\eta} y_\zeta]\,.$$

We say that $(y_\eta)_\eta$ *order converges* (or (o)*-converges*) to $y \in R$ if $y = \limsup_\eta y_\eta = \liminf_\eta y_\eta$, and we write $(o)\lim_{\eta\in\Lambda} y_\eta = y$. An (o)*-net* $(y_\eta)_{\eta\in\Lambda}$ is a monotone decreasing net of elements of $\overline{R}$, such that $\inf_{\eta\in\Lambda} y_\eta = 0$. In particular this defines also the notions of (o)-limit for sequences and of (o)-sequence.

Assumption 2.1. A super Dedekind complete Riesz space R is called an *algebra* if (R, R, R) is a *product triple*, namely if there exists a commutative map $\cdot, \cdot : R \times R \to R$, which we will call *product*, such that the distributive laws and usual compatibility with order hold, and, if $(a_\lambda)_{\lambda\in\Lambda}$ is a net (o)-converging to a, then $\inf_\lambda (a_\lambda \cdot y) = a \cdot y$ for every $y \in R_0^+$.

For example, the Riesz spaces $\mathbb{R}^{\mathbb{N}}$ and $L^0(X, \mathcal{B}, \mu)$ [18, Example 23.3.(iv), pp. 126–12] and [23, p. 70], where $(X, \mathcal{B}, \mu)$ is a measure space with μ positive, σ-additive and σ-finite, are algebras with respect to the usual product.

We remind now the definition and some properties of convex functions.

Definition 2.2. Given a function $f : [a, b] \to R$, we say that f is *convex* in $[a, b]$ if for every $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ and for each $t \in [0, 1]$ we have $f(tx_1 + (1 - t)x_2) \le tf(x_1) + (1 - t)f(x_2)$.

Remark 2.3. Analogously as in the classical case, it is easy to show that a function f is convex if and only if the function

$$\vartheta(t_1, t_2) := \frac{f(t_1) - f(t_2)}{t_1 - t_2}, \quad t_1, t_2 \in [a, b], \quad t_1 \ne t_2,$$

is an increasing function with respect to each variable separately.

Similarly, one can prove the following

Proposition 2.4. *Let $f : [a, b] \to R$ be a convex bounded function and $\delta \in]0, \frac{b-a}{2}[$. Then for every $t_1, t_2 \in [a + \delta, b - \delta]$, with $t_1 \ne t_2$, we have:*

$$\left| \frac{f(t_1) - f(t_2)}{t_1 - t_2} \right| \le \frac{\sup_{x \in [a,b]} f(x) - \inf_{x \in [a,b]} f(x)}{\delta}.$$

3. Global limits

We denote by $\Gamma = (\mathbb{R}^+)^{[a,b]}$, the set of all positive real-valued functions, defined on $[a, b]$, where $a, b \in \mathbb{R}$, $a < b$. We use Γ as a down-directed index set and call the elements of Γ "mesh functions".

From now on, we suppose that R is an algebra, and E denotes a nonempty subset of $[a, b]$.

We consider a R-valued map $\phi = \phi(x, h)$, where $x \in [a, b]$ and $|h|$ is small enough but different from zero. Note that, when we require a condition of the type "$|h| \le \gamma(x)$", with $\gamma \in \Gamma$, we will always suppose that $x + h \in [a, b]$ and/or $x - h \in [a, b]$, without writing it explicitly.

We now introduce the notion of "global" limit (briefly, (g)-limit), which formally lies between pointwise and uniform limits, and in the case $R = \mathbb{R}$ coincides with the pointwise limit.

Definition 3.1. We say that a *global limit* $(g)\lim_{h\to 0} \phi(x, h)$ exists in E and is equal to $\widehat{\phi}(x)$ if

$$\inf_{\gamma \in \Gamma} \left[\sup \left\{ \left| \phi(x, h) - \widehat{\phi}(x) \right| : x \in E, 0 < |h| \le \gamma(x) \right\} \right] = 0.$$

Definition 3.2. We say that $(g)\limsup_{h\to 0} \phi(x, h) = \overline{\phi}(x)$ ($(g)\liminf_{h\to 0} \phi(x, h) = \underline{\phi}(x)$) is a *global limsup* (*global liminf*) in E if there exists an (o)-net $(p_\gamma)_{\gamma \in \Gamma}$ such that for all $\gamma \in \Gamma$ and $x \in E$ we have:

$$0 \le \sup \left\{ \phi(x, h) : 0 < |h| \le \gamma(x) \right\} - \overline{\phi}(x) \le p_\gamma \tag{1}$$
$$\left(0 \le \underline{\phi}(x) - \inf \left\{ \phi(x, h) : 0 < |h| \le \gamma(x) \right\} \le p_\gamma \right).$$

Remark 3.3. Note that the (g)-limit of $\phi(x,h)$ in E exists if and only if $\overline{\phi}(x)=\underline{\phi}(x)$ for every $x\in E$ (see [7]). Furthermore it is readily seen that, if global limits $\overline{\phi}(x)=(g)\limsup_{h\to 0}\phi(x,h)$ and $\underline{\phi}(x)=(g)\liminf_{h\to 0}\phi(x,h)$ exist, then they coincide with the corresponding pointwise limits, that is, for all $x\in$ E

$$\begin{aligned}\overline{\phi}(x)&=\inf_{\gamma\in\Gamma}\left[\sup_{0<|h|\le\gamma(x)}\phi(x,h)\right],\\ \underline{\phi}(x)&=\sup_{\gamma\in\Gamma}\left[\inf_{0<|h|\le\gamma(x)}\phi(x,h)\right].\end{aligned}\tag{2}$$

Indeed, (2) follows from (1) if we take the infimum and the supremum, respectively, as γ varies in Γ.

The following proposition will be useful in the sequel, in particular when we deal with major and minor functions. For the sake of simplicity, we formulate and prove it for the global lim inf. The corresponding statement concerning the global lim sup is analogous.

Proposition 3.4. *Let $f=f(x)$ and $\phi=\phi(x,h)$ be two R-valued maps. We have*

$$(g)\liminf_{h\to 0}\phi(x,h)\ge f(x)\quad \text{in } E$$

if and only if there exists an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that, for all $\gamma\in\Gamma$, for each $x\in E$ and whenever $0<|h|\le\gamma(x)$, the inequality

$$\phi(x,h)\ge f(x)-p_\gamma \tag{3}$$

holds.

Proof. By (3), there exists an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that, for any $\gamma\in\Gamma$ and for every $x\in E$, we have

$$\inf_{0<|h|\le\gamma(x)}\phi(x,h)\ge f(x)-p_\gamma\,.$$

Taking the (o)-limit as γ varies in Γ, we obtain

$$\sup_{\gamma\in\Gamma}\big[\inf_{0<|h|\le\gamma(x)}\phi(x,h)\big]=(o)\lim_{\gamma\in\Gamma}\big[\inf_{0<|h|\le\gamma(x)}\phi(x,h)\big]\ge f(x)$$

for all $x\in E$. In view of Remark 3.3 this proves the sufficiency part.

Conversely, if $\underline{\phi}(x)=(g)\liminf_{h\to 0}\phi(x,h)\ge f(x)$ for any $x\in E$, then there exists an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that, for each $\gamma\in\Gamma$, for every $x\in E$,

$$f(x)-\inf_{0\le|h|\le\gamma(x)}\phi(x,h)\le\underline{\phi}(x)-\inf_{0\le|h|\le\gamma(x)}\phi(x,h)\le p_\gamma\,,$$

and hence $\inf_{0\le|h|\le\gamma(x)}\phi(x,h)\ge f(x)-p_\gamma$. This completes the proof. □

Let Ω be a compact extremally disconnected topological space, such that, thanks to the Maeda–Ogasawara–Vulikh Theorem (see [2, 7]), R is algebraically and lattice isomorphically embedded as an order ideal in the space

$$\mathcal{C}_\infty(\Omega) = \Big\{\varphi : \Omega \to \widetilde{\mathbb{R}} : \varphi \quad \text{is continuous, and} \big\{\omega : |\varphi(\omega)| = +\infty\big\} \\ \text{is nowhere dense in } \Omega\Big\}.$$

Remark 3.5. In what follows we shall often identify an element of R with the corresponding one of $\mathcal{C}_\infty(\Omega)$ obtained by the above embedding, using the same notation for both of them.

We can define a $\widetilde{\mathbb{R}}$-valued map ϕ_ω by setting $\phi_\omega(x,h) := \phi(x,h)(\omega)$ for $x \in [a,b]$ and $h \in \mathbb{R}$. With this notation, as in [7], we have the following

Proposition 3.6. *If a R-valued function $\phi(x,h)$ has (g)-limit in E, then the set $\Omega \setminus W_0$ is meager in Ω, where $W_0 := \{\omega \in \Omega : \lim_{h\to 0} \phi_\omega(x,h)$ exists in $\mathbb{R}$ for every $x \in E\}$. If ϕ is bounded, then the converse holds too. In this case, for every $\omega \in W_0$, we have*

$$\big[(g) \lim_{h\to 0} \phi(x,h)\big](\omega) = \lim_{h\to 0} \phi_\omega(x,h)\,.$$

4. Differential calculus

We now introduce the concepts of "global" continuity and differentiability, using the idea of the global limit.

Definition 4.1. We say that $f : [a,b] \to R$ is (g)-*continuous* in E if there exists an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that, for all $\gamma \in \Gamma$,

$$|f(x+h) - f(x)| \le p_\gamma \quad \text{whenever} \quad \mathrm{x \in E}\,, \quad \mathrm{|h| \le \gamma(x)}\,.$$

For (g)-continuous functions the following result holds [3].

Proposition 4.2. *Let $f : [a,b] \to R$ be a (g)-continuous function in $[a,b]$. Then f is bounded on $[a,b]$.*

Definition 4.3. A function $f : [a,b] \to R$ is (g)-*differentiable in* E if there exists a function $f' : E \to R$ such that

$$\inf_{\gamma\in\Gamma} \left[\sup\left\{ \left|\frac{f(x+h) - f(x)}{h} - f'(x)\right| : x \in E, 0 < |h| \le \gamma(x)\right\}\right] = 0\,.$$

It is easy to see that such a function f' is unique. The function f' will be called the (g)-*derivative* of f, or simply *derivative,* when no confusion can arise.

We observe that, if f is (g)-differentiable with a bounded derivative, then it is (g)-continuous too.

Definition 4.4. A function $f : [a,b] \to R$ is said to be *Lipschitz* in $[a,b]$ if there exists $L \in R^+$ such that $|f(t_1) - f(t_2)| \le L|t_1 - t_2|$ for all t_1, $t_2 \in [a,b]$.

For each R-valued function it is possible to characterize properties of functions f by means of f_ω (see Remark 3.5), for example:

Proposition 4.5 ([7]). *Let $f : [a, b] \to R$ be a function, the following statements hold:*

- *if f is (g)-continuous, then the set $\{\omega \in \Omega : f_\omega$ is not continuous in $[a, b]\}$ is meager in Ω, and, if f is bounded, then the converse holds true;*
- *if f is (g)-differentiable, then $f'(x)(\omega) = {f_\omega}'(x)$ for all $\omega \in \Omega \setminus N^*$ and $x \in [a, b]$, where $N^* := \{\omega \in \Omega : f_\omega$ is not differentiable in $[a, b]\}$ is a meager set independent on x; if f is Lipschitz, then the converse holds true.*

Proposition 4.6. *Let $f : [a, b] \to R$ be a (g)-differentiable function, with derivative bounded in $[a, b]$. Then f is Lipschitz in $[a, b]$.*

Proof. By hypothesis, there exists $L \in R$ such that $|f'(x)| \leq L$ whenever $x \in [a, b]$. Thanks to Proposition 4.5, this implies $|f'_\omega(x)| \leq L(\omega) \in \mathbb{R}$ in the complement of meager subsets of Ω. Thus, by virtue of the mean value theorem, for such ω's the functions f_ω are Lipschitz with Lipschitz constant $L(\omega)$. This means that for every $x_1, x_2 \in [a, b]$ and in the complement of meager subsets of Ω we get:

$$|f_\omega(x_2) - f_\omega(x_1)| \leq L(\omega)|x_2 - x_1|\,,$$

and also

$$|f(x_2) - f(x_1)|(\omega) \leq L(\omega)|x_2 - x_1|\,.$$

So the assertion follows. □

Definition 4.7. A function $f : [a, b] \to R$ is said to be *of bounded variation* in $[a, b]$ if there exists in R the quantity

$$\sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : x_0 = a < x_1 < \cdots < x_n = b \right\}.$$

5. The Riemann and Riemann–Stieltjes integrals

We now recall the definitions of the Riemann integral and the Mengoli–Cauchy integral for Riesz-space-valued functions (see for example [3, 5]).

Definitions 5.1. A *division* of $[a, b]$ is any finite set $\tau = \{x_0, x_1, \ldots, x_n\} \subset [a, b]$, where $x_0 = a$, $x_n = b$ and $x_{i-1} < x_i$ for all $i = 1, \ldots, n$.

A *partition* of $[a, b]$ is a set of the type

$$\mathcal{E} = \big\{(A_i, \xi_i) : i = 1, \ldots, n\big\}\,,$$

where $A_i = [x_{i-1}, x_i]$, $\{x_0, x_1, \cdots, x_n\}$ is a division of $[a, b]$ and $\xi_i \in A_i$ for all $i = 1, \ldots, n$. The quantity $|\mathcal{E}| \equiv \max_i (x_i - x_{i-1})$ is called the *mesh* of a partition $\mathcal{E}$ of $[a, b]$.

Given $\gamma \in \Gamma$, we say that a partition $\mathcal{E}$ of $[a, b]$ is γ*-fine* if $x_i - x_{i-1} \leq \gamma(\xi_i)$ for all $i = 1, \ldots, n$.

Definition 5.2. Given a function $f : [a, b] \to R$ and a partition $\mathcal{E}$ of $[a, b]$, we denote by $S(f, \mathcal{E})$ the sum $\sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1})$ and we call it the *Riemann sum* of f with respect to $\mathcal{E}$.

Definition 5.3. A function $f : [a,b] \to R$ is *Mengoli–Cauchy integrable in* $[a,b]$ if there exists an element $Y \in R$ such that

$$\inf_n \left[\sup \left\{ |S(f,\mathcal{E}) - Y| : |\mathcal{E}| \leq 1/n \right\} \right] = 0 \,, \tag{4}$$

where the symbol $|\mathcal{E}|$ denotes the "mesh" of $\mathcal{E}$.

Definition 5.4. Let R be a Dedekind complete Riesz space, and $f : [a,b] \to R$ a bounded function. We call *upper integral* [resp. *lower integral*] of f the element of R given by

$$\inf_{v \in V_f} \int_a^b v(t)\, dt \qquad \left[\sup_{s \in S_f} \int_a^b s(t)\, dt \right],$$

where

$$V_f \equiv \left\{ v : v \text{ is a step function }, \; v(t) \geq f(t) \text{ for all } t \in [a,b] \right\},$$
$$S_f \equiv \left\{ s : s \text{ is a step function }, \; s(t) \leq f(t) \text{ for all } t \in [a,b] \right\},$$

and the integrals of the involved step functions are understood as in the classical setting. We say that a bounded function $f : [a,b] \to R$ is *Riemann integrable* (or (Ri)-*integrable) in* $[a,b]$, if its lower integral coincides with its upper integral, and, in this case, we call (Ri)-*integral* of f (and write $(Ri) \int_a^b f(t)\, dt$) the common value of them.

Theorem 5.5. [5, Theorems 3.5, 3.6] *A function $f : [a,b] \to R$ is Mengoli–Cauchy integrable if and only if it is Riemann integrable, and in this case the two involved integrals coincide.*

Remark 5.6 (see for example [3,5]). A bounded function $f : [a,b] \to R$ is Riemann integrable if and only if the set $\{\omega \in \Omega : f_\omega$ is not Riemann integrable $\}$ is meager in Ω, and in this case we have, in the complement of meager sets,

$$(Ri) \int_a^b f_\omega(x)\, dx = \left((Ri) \int_a^b f(x)\, dx \right)(\omega) \,.$$

Moreover, if $f : [a,b] \to R$ is (g)-continuous, then it is Riemann integrable.

As in the scalar case, the Torricelli–Barrow theorem and the Fundamental Theorem of Calculus hold:

Theorem 5.7. *Let f be a (g)-continuous function in E. Then its Riemann integral function F, defined by $F(x) = (Ri) \int_a^x f$, is (g)-differentiable in E and*

$$F'(x) = f(x) \,, \quad x \in E \,.$$

Theorem 5.8. *Let $f : [a,b] \to R$ be a (g)-differentiable function, and suppose that its (g)-derivative f' is Riemann integrable. Then*

$$(Ri) \int_a^b f'(t)\, dt = f(b) - f(a) \,.$$

We now turn to the Riemann–Stieltjes integral (for definitions and properties see for example [13]).

Definition 5.9. Given a partition $\mathcal{E} = \{([x_{i-1}, x_i], \xi_i), i = 1, \ldots, n\}$ and $f, g : [a, b] \to R$, we denote by $S_g(f, \mathcal{E})$ the *Riemann sum of f with respect to g,* namely

$$\sum_{i=1}^{n} f(\xi_i)\big[g(x_i) - g(x_{i-1})\big]\,.$$

Definition 5.10. A function $f : [a, b] \to R$ is *Riemann–Stieltjes integrable with respect to* $g : [a, b] \to R$ if there exists an element $Y \in R$ such that

$$\inf_n \Big[\sup\big\{\, |S_g(f, \mathcal{E}) - Y| : \, |\mathcal{E}| \le 1/n\big\}\Big] = 0\,,$$

and in this case we write $(RS) \int_a^b f\, dg = Y$.

The following result will be useful in the sequel (see also [15]).

Proposition 5.11. *Let $f, g : [a, b] \to R$. Suppose that f is (g)-continuous and g is of bounded variation. Then the integral $(RS) \int_a^b f\, dg$ exists in R.*

6. Major and minor functions

In what follows we use the usual notation for the second symmetric difference:

$$\Delta^2 f(x, h) = f(x + h) - 2f(x) + f(x - h)\,.$$

Definition 6.1. Given $f : [a, b] \to R$ and $\emptyset \ne E \subset [a, b]$, we set for $x \in E$

$$\overline{D}^1 f(x) = (g) \limsup_{h\to 0} \frac{f(x + h) - f(x)}{h}\,,$$

$$\underline{D}^1 f(x) = (g) \liminf_{h\to 0} \frac{f(x + h) - f(x)}{h}\,,$$

$$\overline{D}^2 f(x) = (g) \limsup_{h\to 0} \frac{\Delta^2 f(x, h)}{h^2}\,,$$

$$\underline{D}^2 f(x) = (g) \liminf_{h\to 0} \frac{\Delta^2 f(x, h)}{h^2}\,.$$

These quantities are called *upper* and *lower Dini derivatives in E, of order* 1 *and* 2*,* respectively.

It is clear that if $\overline{D}^1 f(x) = \underline{D}^1 f(x)$ then f is (g)-differentiable and the common value coincides with g-derivative of f. If $\overline{D}^2 f(x) = \underline{D}^2 f(x)$ then we call this common value *Riemann (g)-derivative of order* 2 and denote it as $D^2 f(x)$.

Definition 6.2. Given $f : [a, b] \to R$, we say that $G : [a, b] \to R$ is a *major function of order* 1 (resp. 2) for f if it is (g)-continuous in $[a, b]$ (resp. (g)-differentiable in $[a, b]$ with a bounded derivative) and

6.2.1). $G(a) = 0$ (resp. $G(a) = G'(a) = 0$);

6.2.2). there exists a countable set $Q \subset]a, b[$ such that $\underline{D}^1 G(x) \geq f(x)$ ($\underline{D}^2 G(x) \geq f(x)$) for each $x \in]a, b[\setminus Q$.

Let $k = 1, 2$. A function K is a *minor function of order* k *for* f if $-K$ is a major function of order k for $-f$.

We denote by $\mathcal{G}_k$ and $\mathcal{K}_k$ the set of all major and minor functions of order k for f, respectively.

Similar to [4], we get the following results:

Proposition 6.3. *If* $f : [a, b] \to R$ *has both a major function* Ψ *and a minor function* Φ *of order* 1 (*resp.* 2), *then* $\Psi - \Phi$ *is positive, increasing* (*and convex*) *in* $[a, b]$.

Proof. Let Ψ and Φ be any major and minor function of order k for f. Define $T(x) = \Psi(x) - \Phi(x)$ for $x \in [a, b]$. Since Ψ and Φ are (g)-continuous in $[a, b]$, then T is bounded. Thus there exist a nowhere dense set $N^* \subset \Omega$ and an element $L \in R$ such that $|T(x)(\omega)| \leq L(\omega) \in \mathbb{R}$ for all $x \in [a, b]$ and $\omega \in \Omega \setminus N^*$. For each $\omega \in \Omega \setminus N^*$ and $x \in [a, b]$, set $T_\omega(x) := T(x)(\omega)$.

We first consider the case $k = 2$. For every $\omega \in \Omega \setminus N$ and $x \in E =]a, b[\setminus Q$, where $N \supset N^*$ and $Q \subset]a, b[$ are two suitable sets, meager and countable respectively, we have $\underline{D}^2 T(x)(\omega) \geq 0$. Hence

$$\sup_{\gamma \in \Gamma} \left(\inf_{x \in E, 0 < |h| \leq \gamma(x)} \left[\frac{\Delta^2 T(x, h)(\omega)}{h^2} \right] \right) \geq 0\,,$$

that is for all $\omega \in \Omega \setminus N$ and for every $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that for each $x \in E$ and whenever $0 < |h| \leq \gamma(x)$ we get

$$\frac{\Delta^2 T_\omega(x, h)}{h^2} > -\varepsilon\,.$$

Thus there exists $\delta(\omega, \varepsilon, x) > 0$ such that whenever $0 < |h| \leq \delta$ we have

$$\frac{\Delta^2 T_\omega(x, h)}{h^2} > -\varepsilon\,,$$

and then for the usual lower Riemann second symmetric derivative of the real function T_ω we get the inequality $\underline{D}^2 T_\omega(x) \geq 0$, for every $x \in E$ and $\omega \in \Omega \setminus N$. This (see for instance [9]) implies that, for every $\omega \in \Omega \setminus N$, the function T_ω is convex. Fix now x_1, x_2 with $a \leq x_1 < x_2 \leq b$ and $t \in [0, 1]$. Then in the complement of a meager set (independent on the above chosen elements):

$$T_\omega\big(tx_1 + (1 - t)x_2\big) \leq tT_\omega(x_1) + (1 - t)T_\omega(x_2)$$

and hence $T(tx_1 + (1-t)x_2) \leq tT(x_1) + (1-t)T(x_2)$. This proves that T is convex. So (see Remark 2.3) we get:

$$0 = T'(a) \leq \frac{T(x) - T(a)}{x - a} = \frac{T(x)}{x - a} \quad \text{for every } x \in]a, b]\,;$$

$$0 = T'(a) \leq \frac{T(x) - T(y)}{x - y} \quad \text{whenever } x, y \in [a, b],\ x \neq y\,;$$

and thus $T(x) \geq 0$ for every $x \in [a,b]$ and T is increasing in $[a,b]$. This concludes the proof in the case $k = 2$.

In the case $k = 1$, proceeding analogously as above, we get the existence of a meager set $N' \subset \Omega$ and a set $E \subset]a,b[$ such that $]a,b[\setminus E$ is countable, and

$$\liminf_{h\to 0} \frac{T_\omega(x+h) - T_\omega(x)}{h} \geq 0 \quad \text{for every} \quad \omega \in \Omega \setminus \mathrm{N}' \quad \text{and} \quad \mathrm{x} \in \mathrm{E}\,.$$

This implies that T_ω is increasing on $[a,b]$ for such ω's. Thus T is increasing on $[a,b]$ and, since $T(a) = 0$, we get that $T(x) \geq 0$ for all $x \in [a,b]$. This completes the proof. □

Proposition 6.4. *Under the same hypotheses as in Proposition* 6.3, *if* $\emptyset \neq \mathcal{G}_k$, $\emptyset \neq \mathcal{K}_k$ *and* $\sup_{\Psi\in\mathcal{G}_k} \Psi(b) = \inf_{\Phi\in\mathcal{K}_k} \Phi(b)$, *then, for all* $x \in [a,b]$, *we have:*

$$\inf_{\Psi\in\mathcal{G}_k} \Psi(x) = \sup_{\Phi\in\mathcal{K}_k} \Phi(x)\,.$$

Proof. The assertion follows from the fact that $\Psi - \Phi$ is increasing, thanks to Proposition 6.3. □

7. The Perron integral

Definition 7.1. Let $k = 1, 2$. A function $f : [a,b] \to R$ is said to be *Perron integrable of order* k (shortly $\mathcal{P}^k$*-integrable*) in $[a,b]$ if f has both major and minor functions of order k and

$$\inf_{\Psi\in\mathcal{G}_k} \big[\Psi(b)\big] = \sup_{\Phi\in\mathcal{K}_k} \big[\Phi(b)\big] \in R\,.$$

In this case we denote the common value by $(\mathcal{P}^k)\int_a^b f$ or $(\mathcal{P}^k)\int_a^b f(t)\,dt$. By Proposition 6.4 the function $I_k(x) = \inf_{\Psi\in\mathcal{G}_k}[\Psi(x)] = \sup_{\Phi\in\mathcal{K}_k}[\Phi(x)]$ is well-defined for each $x \in [a,b]$.

We now turn to some fundamental properties of $\mathcal{P}^k$-integral.

Proposition 7.2. *If* $f, f_1, f_2 : [a,b] \to R$ *are* $\mathcal{P}^k$*-integrable for* $k = 1, 2$ *and* c_1, $c_2 \in \mathbb{R}$, *then* $c_1 f_1 + c_2 f_2$ *is* $\mathcal{P}^k$*-integrable too, and we have:*

- **i)** $(\mathcal{P}^k)\int_a^b (c_1 f_1 + c_2 f_2) = c_1(\mathcal{P}^k)\int_a^b f_1 + c_2(\mathcal{P}^k)\int_a^b f_2$ (*linearity*).
- **ii)** $(\mathcal{P}^k)\int_a^b f_1 \leq (\mathcal{P}^k)\int_a^b f_2$ *if* $f_1 \leq f_2$ (*monotonicity*).
- **iii)** *Moreover* f *is* $\mathcal{P}^k$*-integrable in every subinterval* $[a',b'] \subset [a,b]$. *If* $a < c < b$, *then*
$$(\mathcal{P}^k)\int_a^b f = (\mathcal{P}^k)\int_a^c f + (\mathcal{P}^k)\int_c^b f\,. \tag{5}$$
- **iv)** *Conversely, if* $a < c < b$ *and* f *is* $\mathcal{P}^k$*-integrable both in* $[a,c]$ *and in* $[c,b]$, *then it is* $\mathcal{P}^k$*-integrable in* $[a,b]$ *too, and the formula* (5) *holds.*

Proof. The proof is similar to the one in the real-valued case (see [10]). □

Proposition 7.3. *Let $k = 1,2$. If f is $\mathcal{P}^k$-integrable in $[a,b]$ then, for every $x \in [a,b]$,*

$$(\mathcal{P}^k) \int_a^x f(t)\, dt = I_k(x)\,,$$

where I_k is as in Definition 7.1.

Proof. It follows from the integrability in subintervals that $\inf_{\Psi \in \mathcal{G}_k^{[a,x]}} \Psi(x) = \sup_{\Phi \in \mathcal{K}_k^{[a,x]}} \Phi(x) \in R$ for all $x \in [a,b]$, where the symbols $\mathcal{G}_k^{[a,x]}$ and $\mathcal{K}_k^{[a,x]}$ denote the classes of all major and minor functions of order k for f in the interval $[a,x]$. But, if $\Psi \in \mathcal{G}_k$ and $\Phi \in \mathcal{K}_k$ (relative to the interval $[a,b]$), then their restrictions $\Psi_{|[a,x]} \in \mathcal{G}_k^{[a,x]}$, $\Phi_{|[a,x]} \in \mathcal{K}_k^{[a,x]}$, and hence we have:

$$\inf_{\Psi \in \mathcal{G}_k^{[a,x]}} \Psi(x) \le \inf_{\Psi \in \mathcal{G}_k} \Psi_{|[a,x]}(x) = \sup_{\Phi \in \mathcal{K}_k} \Phi_{|[a,x]}(x) \le \sup_{\Phi \in \mathcal{K}_k^{[a,x]}} \Phi(x) = \inf_{\Psi \in \mathcal{G}_k^{[a,x]}} \Psi(x)\,.$$

This implies the assertion. □

We call I_k the $\mathcal{P}^k$*-integral function* associated to f, with the value $I_k(b)$ being the $\mathcal{P}^k$*-integral of* f in $[a,b]$.

Remark 7.4. Note that

$$\inf_{\Psi \in \mathcal{G}_k} \left(\sup_{x \in [a,b]} \big(\Psi(x) - I_k(x)\big) \right) = \inf_{\Phi \in \mathcal{K}_k} \left(\sup_{x \in [a,b]} \big(I_k(x) - \Phi(x)\big) \right) = 0\,.$$

Proposition 7.5. *If $K \in \mathcal{K}_1$, $G \in \mathcal{G}_1$ (resp. $\Phi \in \mathcal{K}_2, \Psi \in \mathcal{G}_2$), then $I_1 - K$ and $G - I_1$ (resp. $I_2 - \Phi$ and $\Psi - I_2$) are R_0^+-valued increasing (and resp. convex) functions.*

Proof. We prove only convexity of $I_2 - \Phi$, since the proof of the other properties is analogous. As R is super Dedekind complete, there exists a sequence $(\Psi_n)_n$ in $\mathcal{G}_2$ such that $\inf_n \Psi_n(b) = I_2(b)$. Fix now $\overline{x} \in [a,b]$. Since $0 \le \Psi_n(\overline{x}) - I_2(\overline{x}) \le \Psi_n(b) - I_2(b)$, it follows that $(o) \lim_n \Psi_n(\overline{x}) = I_2(\overline{x})$. Choose now arbitrarily $\Phi \in \mathcal{K}_2$. Then $I_2 - \Phi$ is the (o)-limit of $\Psi_n - \Phi$, and thus $I_2 - \Phi$ turns out to be a convex map, being the (o)-limit of a sequence of convex functions. □

Proposition 7.6. *Let $k = 1,2$. If f is $\mathcal{P}^k$-integrable, then I_k is (g)-continuous.*

Proof. First of all, we observe that, thanks to Propositions 3.6, 4.5 and 4.6, if G (resp. K) is a major (minor) function of order k ($k = 1,2$) for f, then in the complement of meager subsets of Ω we get that G_ω (K_ω) is a major (minor) function of order k for f_ω. We note that, by construction, for $k = 1,2$ G and K are bounded, and for $k = 2$ G and K are Lipschitz by Proposition 4.6. From this, by virtue of the Maeda–Ogasawara–Vulikh theorem it follows that, if $f : [a,b] \to R$ is $\mathcal{P}^k$-integrable ($k = 1,2$), then the f_ω's are real-valued and $\mathcal{P}^k$-integrable in the complement of a meager subset $N^* \subset \Omega$, and, if I_k and $I_{\omega,k}$ are the $\mathcal{P}^k$-integral functions associated with f and f_ω respectively, then we have $I_{\omega,k}(x) = I_k(x)(\omega)$ for all $x \in [a,b]$ and in the complement of a meager subset $N' \subset \Omega$. Without loss of generality, we can suppose that $N' \supset N^*$.

Let now $\omega \in \Omega \setminus N'$ and $k = 1, 2$. Then $I_{\omega,k}$ is continuous as $\mathcal{P}^k$-integral function of the real-valued function f_ω. Since by construction I_k is bounded, then by Proposition 4.5 we get (g)-continuity of I_k. This completes the proof. □

Proposition 7.7. *Every (Ri)-integrable function $f : [a, b] \to R$ is $\mathcal{P}^1$-integrable, and*

$$(Ri) \int_a^x f(t)\, dt = I_1(x) \quad \textit{for all} \quad x \in [a, b]\,.$$

Proof. Let $v \in V_f$, $s \in S_f$ (see Definition 5.4), and put $T_v(x) := \int_a^x v(t)\, dt$, $T_s(x) := \int_a^x s(t)\, dt$. Then, by Proposition 7.6, T_v and T_s are (g)-continuous, and moreover, by proceeding with similar arguments as in the Torricelli–Barrow theorem, we get that $\underline{D}^1 T_v(x) \geq f(x) \geq \overline{D}^1 T_s(x)$ in the complement of a finite set of x's belonging to $[a, b]$, depending on the involved step functions v and s. So, thanks to super Dedekind completeness of R and Riemann integrability of f, there exist two sequences $(v_n)_n$ and $(s_n)_n$ in V_f and S_f, respectively, which can be supposed, without loss of generality, to be equibounded, and such that

$$0 = \inf_{s \in S_f} \int_a^b v(t)\, dt - \sup_{s \in S_f} \int_a^b s(t)\, dt = \inf_n \int_a^b v_n(t)\, dt - \sup_n \int_a^b s_n(t)\, dt\,. \tag{6}$$

The functions T_{v_n} and T_{s_n}, $n \in \mathbb{N}$, turn out to be a major and a minor function for f, respectively. Thus, taking into account (6) and using Riemann integrability of f, we get:

$$0 \leq \inf_{G \in \mathcal{G}_1} G(b) - \sup_{K \in \mathcal{K}_1} K(b) \leq \inf_n \int_a^b v_n(t)\, dt - \sup_n \int_a^b s_n(t)\, dt \tag{7}$$

and so, from (7) we have

$$\inf_{G \in \mathcal{G}_1} G(b) - \sup_{K \in \mathcal{K}_1} K(b) = 0\,,$$

getting $\mathcal{P}^1$-integrability of f in $[a, b]$. Moreover, from Proposition 7.2 iii) we get $\mathcal{P}^1$-integrability of f in every interval $[a, x]$, with $a < x \leq b$. The conclusion follows by Proposition 7.3. □

We now compare the Perron integral of order 1 with the one of order 2.

Theorem 7.8. *If $f : [a, b] \to R$ is $\mathcal{P}^1$-integrable, then f is $\mathcal{P}^2$-integrable too. Moreover we have:*

$$I_2(b) = \left(\mathcal{P}^2\right) \int_a^b f = (\mathcal{P}^1) \int_a^b \left[(\mathcal{P}^1) \int_a^x f(t)\, dt \right] dx\,.$$

Proof. Let G be any major function of order 1 for f in $[a, b]$. We prove that its $\mathcal{P}^1$-integral function Ψ is a major function of order 2 for f. We note first that, by definition, $\Psi(a) = 0$. Moreover, since G, by hypothesis, is (g)-continuous, then, by Torricelli–Barrow theorem, Ψ is (g)-differentiable and $\Psi'(x) = G(x)$ for every $x \in [a, b]$, and hence $\Psi'(a) = 0$, since, by hypothesis, $G(a) = 0$.

Proceeding similarly as in [4], we now prove the second property of major functions of the order 2. First of all, we observe that by the same property related to the major function of the order 1, Proposition 3.4 implies the existence of an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ and a set $E \subset]a,b[$ such that $]a,b[\setminus E$ is countable and, for every $\gamma \in \Gamma$ and $x \in E$ with $0 < |h| \le \gamma(x)$, we have

$$\frac{G(x+h)-G(x)}{h} \ge f(x) - p_\gamma\,.$$

In particular, when $t > 0$, we get

$$G(x+t)-G(x) \ge t\,f(x) - t\,p_\gamma\,; \quad G(x-t)-G(x) \le -t\,f(x) + t\,p_\gamma\,.$$

Taking the Riemann integral, we get, whenever $x \in E$ and $0 < h \le \gamma(x)$:

$$\begin{aligned}(Ri)\int_0^h \big[G(x+t)-G(x)\big]\,dt &= \Psi(x+h)-\Psi(x)-h\,G(x)\\ &\ge \frac{h^2}{2}\,f(x) - \frac{h^2}{2}\,p_\gamma\,,\\ -(Ri)\int_0^h \big[G(x-t)-G(x)\big]\,dt &= \Psi(x-h)-\Psi(x)+h\,G(x)\\ &\ge \frac{h^2}{2}\,f(x) - \frac{h^2}{2}\,p_\gamma\,,\end{aligned}$$

and so

$$\frac{\Delta^2\Psi(x,h)}{h^2} \ge f(x) - p_\gamma\,.$$

It is easy to see that the last inequalities hold also with $h < 0$. By Proposition 3.4, we get that $\underline{D}^2\Psi(x) \ge f(x)$ for all $x \in E$. In the same way we prove that, if K is a minor function of order 1 for f, then its $\mathcal{P}^1$-integral function Φ is a minor function of order 2 for f. This implies obviously that f is $\mathcal{P}^2$-integrable and $(\mathcal{P}^2)\int_a^b f = (\mathcal{P}^1)\int_a^b I_1$. $\square$

For the converse we prove

Theorem 7.9. *If $f : [a,b] \to R$ is $\mathcal{P}^2$-integrable, then I_2' is $\mathcal{P}^1$-integrable. Moreover, if Ψ (resp. Φ) is a major (minor) function of order 2 for f, then*

$$\Psi'(x) - I_2'(x) \ge 0 \quad \big(I_2'(x) - \Phi'(x) \ge 0\big) \quad \textit{for all} \quad x \in [a,b].$$

Proof. In order to prove the first part of the theorem, it is enough to check that I_2 is both a major and a minor function of order 1 of I_2'. First of all, we note that $I_2(a) = 0$. Moreover, observe that I_2 is Lipschitz in $[a,b]$: indeed, every $\Psi \in \mathcal{G}_2$ and $\Phi \in \mathcal{K}_2$ has by hypothesis bounded (g)-derivatives in $[a,b]$, and hence is Lipschitz in $[a,b]$, thanks to Proposition 4.6. So, since $\Psi - I_2$ and $I_2 - \Phi$ are increasing in

$[a, b]$, there exist two elements L_Ψ, $l_\Phi \in R$ such that, for all t_1, $t_2 \in [a, b]$ with $t_1 \neq t_2$, we get:

$$L_\Psi \geq \frac{\Psi(t_2) - \Psi(t_1)}{t_2 - t_1} \geq \frac{I_2(t_2) - I_2(t_1)}{t_2 - t_1} \geq \frac{\Phi(t_2) - \Phi(t_1)}{t_2 - t_1} \geq l_\Phi\,;$$

Lipschitzianity of I_2 follows from these inequalities.

Let now $I_{\omega,2}$ be the integral function associated with f_ω: we prove that $I_{\omega,2}$ is differentiable in $[a, b]$ for every $\omega \in \Omega \setminus N$, where N is a suitable meager set. For $\omega \notin N$, let $\mathcal{G}_{\omega,2}$ and $\mathcal{K}_{\omega,2}$ be the classes of all major and minor functions of order 2 for f_ω respectively. Fix now arbitrarily $\omega \in \Omega \setminus N$, $\varepsilon > 0$ and $0 < \delta < \frac{b-a}{2}$: then there exists $\Phi^{(\omega)} \in \mathcal{K}_{\omega,2}$ such that, for every $x, x_0 \in [a, b]$, $0 \leq |(I_{\omega,2} - \Phi^{(\omega)})(x) - (I_{\omega,2} - \Phi^{(\omega)})(x_0)| \leq \varepsilon$, and hence, whenever x and x_0 are taken in $[a + \delta, b - \delta]$ with $x \neq x_0$, by Proposition 2.4 we have:

$$0 \leq \frac{(I_{\omega,2} - \Phi^{(\omega)})(x) - (I_{\omega,2} - \Phi^{(\omega)})(x_0)}{x - x_0} \leq \frac{\varepsilon}{\delta}\,.$$

We get:

$$\begin{aligned}
\liminf_{x \to x_0} \frac{I_{\omega,2}(x) - I_{\omega,2}(x_0)}{x - x_0} &\leq \limsup_{x \to x_0} \frac{I_{\omega,2}(x) - I_{\omega,2}(x_0)}{x - x_0} \\
&\leq \limsup_{x \to x_0} \frac{(I_{\omega,2} - \Phi^{(\omega)})(x) - (I_{\omega,2} - \Phi^{(\omega)})(x_0)}{x - x_0} \\
&\quad + (\Phi^{(\omega)})'(x_0) \\
&= \frac{\varepsilon}{\delta} + \lim_{x \to x_0} \frac{\Phi^{(\omega)}(x) - \Phi^{(\omega)}(x_0)}{x - x_0} \\
&\leq \frac{\varepsilon}{\delta} + \liminf_{x \to x_0} \frac{I_{\omega,2}(x) - I_{\omega,2}(x_0)}{x - x_0}\,,
\end{aligned}$$

because, by Proposition 7.5, $I_{\omega,2} - \Phi^{(\omega)}$ is increasing in $[a, b]$. From this, by arbitrariness of ε, it follows that $I_{\omega,2}$ is differentiable at x_0. Hence, by the arbitrariness of δ, we get the differentiability of $I_{\omega,2}$ in $]a, b[$. Let now $\Psi^{(\omega)} \in \mathcal{G}_{\omega,2}$ and $\Phi^{(\omega)} \in \mathcal{K}_{\omega,2}$. Since $\Psi^{(\omega)} - I_{\omega,2}$ and $I_{\omega,2} - \Phi^{(\omega)}$ are increasing in $[a, b]$, we get, for all $x, x_0 \in [a, b]$ with $x \neq x_0$:

$$\frac{\Psi^{(\omega)}(x) - \Psi^{(\omega)}(x_0)}{x - x_0} \geq \frac{I_{\omega,2}(x) - I_{\omega,2}(x_0)}{x - x_0} \geq \frac{\Phi^{(\omega)}(x) - \Phi^{(\omega)}(x_0)}{x - x_0}\,. \tag{8}$$

As $\Psi^{(\omega)}(a) = \Phi^{(\omega)}(a) = 0$, it follows from (8), with $x_0 = a$, that ${I_{\omega,2}}'(a) = 0$. Furthermore, we observe that, since every function $\Phi^{(\omega)} \in \mathcal{K}_{\omega,2}$ is differentiable at b and is such that $I_{\omega,2} - \Phi^{(\omega)}$ is convex in $[a, b]$ (see also Proposition 7.5), then the limit

$$\lim_{x \to b^-} \frac{I_{\omega,2}(x) - I_{\omega,2}(b)}{x - b}$$

exists in $\overline{\mathbb{R}}$. Differentiability of $I_{\omega,2}$ at the point b follows from this and (8) used with $x_0 = b$. Thus, by Proposition 4.5, it follows that I_2 is (g)-differentiable in

$[a, b]$ and ${I_2}'(a) = 0$. Hence, I_2 is both a major and a minor function of order 1 of ${I_2}'$.

Now, to prove the last part of the theorem, let Ψ be an arbitrary major function of order 2 for f. We observe that, taking into account the convexity of $\Psi - I_2$ and the equality $\Psi(a) = \Psi'(a) = I_2(a) = {I_2}'(a) = 0$, we get $\Psi'(x) - {I_2}'(x) \geq 0$ for all $x \in [a, b]$. An analogous result holds for minor functions of order 2 for f. This completes the proof. □

Remark 7.10. It is not difficult to construct an example showing that our $\mathcal{P}^2$-integral is essentially more general than $\mathcal{P}^1$-integral. It is enough, in the case $R = \mathbb{R}$, to consider $\Psi(x) = x^2 \sin \frac{1}{x}$. Then $f(x) = D^2\Psi(x)$ exists everywhere on $[-1, 1]$ and is $\mathcal{P}^2$-integrable with $\Psi(x) - \Psi(-1)$ being its $\mathcal{P}^2$-integral function on $[-1, 1]$, but f is not $\mathcal{P}^1$-integrable on the same interval because it has no continuous $\mathcal{P}^1$-integral function (note that $\Psi'(x)$ is bounded on $[-1, 1]$ but it is not continuous at $x = 0$).

8. Integration by parts

Here we consider separately the cases $k = 1$ and $k = 2$ using the Riemann–Stieltjes integral in the formula for the case $k = 1$. Then this formula will be used to obtain the second order result.

8.1. The integration by parts formula for the $\mathcal{P}^1$-integral

We now prove the main theorem in the case $k = 1$.

Theorem 8.1. *Let $f, g : [a, b] \to R$, suppose that g is Lipschitz and f is $\mathcal{P}^1$-integrable in $[a, b]$. Then $f \cdot g$ is $\mathcal{P}^1$-integrable in $[a, b]$ too, and we have*

$$(\mathcal{P}^1) \int_a^b f \cdot g = I_1(b)g(b) - (RS) \int_a^b I_1 \, dg \,,$$

where I_1 is the $\mathcal{P}^1$-integral function associated to f.

Proof. Since g is Lipschitz, then g is of bounded variation, and hence g is the difference of two monotone increasing functions. So, without loss of generality, we can suppose that $g(a) = 0$ and $g(t_1) \leq g(t_2)$ whenever $a \leq t_1 < t_2 \leq b$. In some cases below we shall apply Proposition 3.4 without mentioning it explicitly.

Let G and K be any major and minor function of order 1 for f. We set for $x \in [a, b]$

$$M(x) = G(x)g(x) - (RS) \int_a^x G \, dg \,, \quad m(x) = K(x)g(x) - (RS) \int_a^x K \, dg \,. \tag{9}$$

For every $x \in]a, b[$ and $h \neq 0$, we get:

$$\begin{aligned} \frac{G(x+h)g(x+h) - G(x)g(x)}{h} &= G(x+h)\frac{g(x+h) - g(x)}{h} \\ &\quad + \frac{G(x+h) - G(x)}{h} g(x) \,, \end{aligned}$$

and so

$$\frac{M(x+h)-M(x)}{h} = \frac{G(x+h)-G(x)}{h} g(x) + \frac{1}{h}(RS)\int_x^{x+h} \big[G(x+h)-G(t)\big]\, dg(t)\,.$$

Moreover, by (g)-continuity of G, there exists an (o)-net $(p_\gamma)_{\gamma\in\Gamma}$ such that $|G(x+h)-G(t)| \le p_\gamma$ for every $\gamma \in \Gamma$ and whenever $x \in]a,b[$, $0 < h \le \gamma(x)$ and $t \in [x, x+h]$. Then

$$\begin{aligned}\left|\frac{1}{h}(RS)\int_x^{x+h} \big[G(x+h)-G(t)\big] dg(t)\right| &\le \frac{1}{h}(RS)\int_x^{x+h} |G(x+h)-G(t)|\, dg(t)\\ &\le \frac{1}{h}\, p_\gamma (RS)\int_x^{x+h} dg(t)\\ &\le p_\gamma \left|\frac{g(x+h)-g(x)}{h}\right| \le p_\gamma\, L\,,\end{aligned}$$

where $L \in R^+$ is a Lipschitz constant for g; we get an analogous result for $-\gamma(x) \le h < 0$. Thus, since G is a major function of order 1 for f, we have the existence of a set $E \subset]a,b[$ such that $]a,b[\setminus E$ is countable and, for every $\gamma \in \Gamma$ and $x \in E$,

$$\begin{aligned}&\inf\left\{\frac{M(x+h)-M(x)}{h} : x \in E, 0 < |h| \le \gamma(x)\right\}\\ &\quad \ge \left[\inf\left\{\frac{G(x+h)-G(x)}{h} : x \in E, 0 < |h| \le \gamma(x)\right\}\right] \cdot g(x) - p_\gamma L\\ &\quad\ge f(x)g(x) - p_\gamma L\,.\end{aligned}$$

From this it follows that

$$\underline{D}^1 M(x) \ge f(x)\, g(x) \quad \text{for all} \quad x \in E\,.$$

Moreover, we observe that M is (g)-continuous, because so are G and g, and the integral function $G_g(x) := (RS)\int_a^x G\, dg$ is Lipschitz (this is a consequence of boundedness of G and Lipschitzianity of g). Furthermore, $M(a) = 0$, since $G(a) = 0$. Thus, M is a major function of order 1 for $f \cdot g$. Similarly, it is possible to check that the function m defined in (9) is a minor function of order 1 for $f \cdot g$. Finally, since $G - I_1$ is increasing, for every $x \in [a,b]$ we get:

$$\begin{aligned}0 &\le (RS)\int_a^x \Big\{G(x) - I_1(x) - \big[G(t) - I_1(t)\big]\Big\}\, dg(t)\\ &= \big[G(x) - I_1(x)\big]\, g(x) - (RS)\int_a^x \big[G(t) - I_1(t)\big]\, dg(t)\\ &= G(x)g(x) - (RS)\int_a^x G\, dg - I_1(x)g(x) + (RS)\int_a^x I_1\, dg \qquad (10)\\ &= M(x) - I_1(x)g(x) + (RS)\int_a^x I_1\, dg\,.\end{aligned}$$

We now denote by $\mathcal{M}$ and Ξ the classes of all major and minor functions of order 1 for $f\cdot g$ and by $\mathcal{M}_1$ the class of that major functions of order 1 for $f\cdot g$ in $[a,b]$ which satisfies (9). By properties of the major functions G of order 1 for f and (10), we have, for all $x\in[a,b]$:

$$\begin{aligned}0 &\le \inf_{M\in\mathcal{M}}\left(\sup_{x\in[a,b]}\left(M(x)-I_1(x)g(x)+(RS)\int_a^x I_1dg\right)\right)\\ &\le \inf_{M\in\mathcal{M}_1}\left(\sup_{x\in[a,b]}\left(M(x)-I_1(x)g(x)+(RS)\int_a^x I_1dg\right)\right)\\ &= \inf_{G\in\mathcal{G}_1}\left(\sup_{x\in[a,b]}\left((RS)\int_a^x\Big\{G(x)-I_1(x)-\big[G(t)-I_1(t)\big]\Big\}\,dg(t)\right)\right)=0\,,\end{aligned}$$

since $\inf_{G\in\mathcal{G}_1}(\sup_{x\in[a,b]}[G(x)-I_1(x)])=0$, where $\mathcal{G}_1$ is, as before, the class of all major functions of order 1 for f.

Analogously we get, for each $x\in[a,b]$:

$$\inf_{m\in\Xi}\left(\sup_{x\in[a,b]}\left(I_1(x)g(x)-(RS)\int_a^x I_1\,dg-m(x)\right)\right)=0\,.$$

Thus, we have proved that $f\cdot g$ is $\mathcal{P}^1$-integrable and

$$(\mathcal{P}^1)\int_a^x f\cdot g=I_1(x)\,g(x)-(RS)\int_a^x I_1\,dg\,,\quad x\in[a,b]\,. \qquad \square$$

Remark 8.2. In the classical integration by part formula for the Perron integral in the real-valued case the multiplier is a function of bounded variation. Whether the Lipschitz function g can be replaced with a function of bounded variation in the above theorem, is an open problem for the moment.

8.2. The formula of integration by parts for the $\mathcal{P}^2$-integral

We begin with some lemmas. As in the scalar case (see for example [20, 21]), the following lemma holds.

Lemma 8.3. *Let $f,g:[a,b]\to R$, $x\in[a,b]$, $h\in\mathbb{R}\setminus\{0\}$. Then we get:*

$$\begin{aligned}\frac{\Delta^2(f\cdot g)(x,h)}{h^2} &= f(x+h)\frac{\Delta^2 g(x,h)}{h^2}\\ &\quad+2\frac{f(x+h)-f(x)}{h}\;\frac{g(x)-g(x-h)}{h}\\ &\quad+\frac{\Delta^2 f(x,h)}{h^2}\,g(x-h)\,.\end{aligned}$$

The proof follows by simple computation.

Lemma 8.4. *Let $p,u,v,w,y,z\in R$ be such that $p,w\ge 0$; $u\ge v-p$, $z\ge 0$, $0\ge z-y\ge -w$. Then*

$$u\,z\ge v\,y-v^+\,w-p\,y\,. \tag{11}$$

Proof. Since $z \geq 0$ and $u \geq v - p$, then we get

$$u\,z \geq v\,z - p\,z = v^+\,z - v^-\,z - p\,z\,.$$

Moreover, as $z \geq y - w$, we have $v^+\,z \geq v^+\,y - v^+\,w$. Furthermore, since $z \leq y$, we get $-v^-\,z \geq -v^-\,y$, $-p\,z \geq -p\,y$, and hence

$$u\,z \geq v^+\,y - v^+\,w - v^-\,y - p\,y = v\,y - v^+\,w - p\,y\,. \qquad \square$$

Theorem 8.5. *Let $f : [a,b] \to R$ be $\mathcal{P}^2$-integrable in $[a,b]$ and let $g : [a,b] \to R$ be (g)-differentiable in $[a,b]$, with the derivative g' being Lipschitz in $[a,b]$. Then $f \cdot g$ is $\mathcal{P}^2$-integrable in $[a,b]$, and*

$$(\mathcal{P}^2)\int_a^b f \cdot g + (\mathcal{P}^1)\int_a^b \left[(\mathcal{P}^1)\int_a^x {I_2}'(t)g'(t)dt\right]dx = I_2(b)g(b) - (\mathcal{P}^1)\int_a^b I_2 \cdot g'\,.$$

Proof. Without loss of generality, we can suppose $g(x) \geq 0$ and g increasing in $[a,b]$, so that $g'(x) \geq 0$ for every $x \in [a,b]$. Let Ψ be any major function of order 2 for f. We prove that

$$S(x) = \Psi(x)g(x) - (\mathcal{P}^1)\int_a^x \Psi \cdot g'\,, \quad x \in [a,b] \tag{12}$$

is a major function of order 2 for $f \cdot g + {I_2}' \cdot g'$, where I_2 is the integral function of order 2 associated with f.

It is easy to check that all rules of differentiation of the product hold, as in the classical case. From this and Theorem 5.7 we have:

$$S'(x) = \Psi'(x)\,g(x) + \Psi(x)\,g'(x) - \Psi(x)\,g'(x) = \Psi'(x)\,g(x) \quad \text{forall} \quad \mathrm{x} \in [\mathrm{a},\mathrm{b}]\,;$$
$$S(a) = S'(a) = 0\,.$$

Fix now $x \in [a,b]$, and put $\overline{\Psi}(t) = \overline{\Psi}_x(t) = \Psi(t) - P(t)$, where $P(t) = \Psi(x) + (t-x)\Psi'(x)$ and set

$$\overline{S}(t) = \overline{S}_x(t) = \overline{\Psi}(t)g(t) - (\mathcal{P}^1)\int_a^t \overline{\Psi} \cdot g', \quad t \in [a,b]\,.$$

Note that

$$\Delta^2\Psi(x,h) = \Delta^2\overline{\Psi}(x,h)\,. \tag{13}$$

We also have:

$$(S - \overline{S})'(t) = P'(t)\,g(t) + P(t)\,g'(t) - P(t)\,g'(t) = \Psi'(x)\,g(t)\,, \quad t \in [a,b]\,.$$

Let $h > 0$. We have: $\Delta^2 S(x,h) = \Delta^2\overline{S}(x,h) + \Delta^2(S - \overline{S})(x,h)$. It is easy to check that

$$\Delta^2(S - \overline{S})(x,h) = \Psi'(x)\left((\mathcal{P}^1)\int_0^h - (\mathcal{P}^1)\int_{-h}^0\right)\left[(\mathcal{P}^1)\int_0^t \big(g'(x+u) - g'(x)\big)du\right]dt$$
$$+ \Psi'(x)g'(x)h^2\,.$$

Let L^* be such that $|\Psi'(x)| \le L^*$ for all $x \in [a,b]$, and $0 \le M^* \in R$ be a Lipschitz constant for g'. We have: $|g'(x+u) - g'(x)| \le u\, M^*$, and then

$$\frac{\Delta^2 (S - \overline{S})(x,h)}{h^2} \ge -\frac{L^*\, M^*\, h}{3} + \Psi'(x) g'(x)\,. \tag{14}$$

To estimate $\frac{\Delta^2 \overline{S}(x,h)}{h^2}$ we use Lemma 8.3 and (13) to obtain, for all $x \in]a,b[$:

$$\begin{aligned}\frac{\Delta^2 \overline{S}(x,h)}{h^2} \ge\ & \overline{\Psi}(x+h)\frac{\Delta^2 g(x,h)}{h^2} + 2\frac{\overline{\Psi}(x+h) - \overline{\Psi}(x)}{h}\,\frac{g(x) - g(x-h)}{h} \\ & + \frac{\Delta^2 \Psi(x,h)}{h^2}\, g(x-h) \\ & - \frac{1}{h^2}\left((\mathcal{P}^1)\int_x^{x+h} - (\mathcal{P}^1)\int_{x-h}^{x} \right) \overline{\Psi}(\tau)\, g'(\tau)\, d\tau\,. \end{aligned} \tag{15}$$

We note first that according to the (g)-differentiability of Ψ there exists an (o)-net $(z_\gamma)_{\gamma\in\Gamma}$ such that, for every $\gamma \in \Gamma$, whenever $x, \tau \in [a,b]$, $0 < |\tau - x| \le \gamma(x)$,

$$|\overline{\Psi}(\tau)| = |\Psi(\tau) - \Psi(x) - (\tau - x)\Psi'(x)| \le |\tau - x|\, z_\gamma\,. \tag{16}$$

Moreover, there exists a positive element $K_0 \in R$ such that, for every $\tau \in [a,b]$, $|g'(\tau)| \le K_0$. Using this, we obtain:

$$\begin{aligned} \left|(\mathcal{P}^1)\int_x^{x+h} \overline{\Psi}(\tau)\, g'(\tau)\, d\tau\right| &= \left|(\mathcal{P}^1)\int_x^{x+h} \big[\Psi(\tau) - \Psi(x) - (\tau - x)\Psi'(x)\big] g'(\tau)\, d\tau\right| \\ &\le \left[(\mathcal{P}^1)\int_x^{x+h} (\tau - x) d\tau\right] K_0\, z_\gamma = \frac{h^2}{2}\, K_0\, z_\gamma \end{aligned} \tag{17}$$

whenever $x \in]a,b[$ and $0 < h \le \gamma(x)$. Analogously we prove that for such x's and h's we have:

$$\left|(\mathcal{P}^1)\int_{x-h}^{x} \overline{\Psi}(\tau)\, g'(\tau)\, d\tau\right| \le \frac{h^2}{2}\, K_0\, z_\gamma\,. \tag{18}$$

Now, using the Maeda–Ogasawara–Vulikh Theorem, we set $g_\omega(x) = g(x)(\omega)$ for all $x \in [a,b]$ and $\omega \in \Omega$ (see Remark 3.5). Let $L \in R^+$ be a Lipschitz constant for g'. There exists a meager set $N \subset \Omega$ such that, for each $\omega \in \Omega \setminus N$, for any $x \in]a,b[$ and $h > 0$, we have, for a suitable point $\xi_{h,\omega}$ belonging to the interval $[x, x+h]$,

$$\left|\frac{\Delta^2 g_\omega(x,h)}{h^2}\right| = \left|\frac{g'_\omega(\xi_{h,\omega}) - g'_\omega(\xi_{h,\omega} - h)}{h}\right| \le L(\omega) \in \mathbb{R}\,,$$

by virtue of the mean value theorem applied to the function $J^{(\omega)}(t) := g_\omega(t) - g_\omega(t-h)$. From this it follows that

$$\left|\frac{\Delta^2 g(x,h)}{h^2}\right| \le L\,. \tag{19}$$

Now once again using the (g)-differentiability of Ψ we take, the (o)-net $(z_\gamma)_{\gamma\in\Gamma}$ for which (16) holds having also in mind that $|\Psi'(x)| \le L^*$ for all $x \in [a,b]$, we get for $x \in]a,b[$ and $0 < h \le \gamma(x)$:

$$\left|\overline{\Psi}(x+h)\right| = |\Psi(x+h) - \Psi(x) - h\Psi'(x)| \le h z_\gamma\,; \tag{20}$$
$$\left|\frac{\overline{\Psi}(x+h) - \overline{\Psi}(x)}{h}\right| = \left|\frac{\Psi(x+h) - \Psi(x)}{h} - \Psi'(x)\right| \le z_\gamma\,.$$

Moreover, since g' is Lipschitz, then g' is (g)-continuous and hence bounded too. Thus, by Proposition 4.6, g is Lipschitz and (g)-continuous too. So there exists an (o)-net $(w_\gamma)_{\gamma\in\Gamma}$ such that, for every $\gamma \in \Gamma$ and whenever $x \in [a,b]$ with $0 < h \le \gamma(x)$, we have $0 \ge g(x-h) - g(x) \ge -w_\gamma$.

Let now $C \in R^+$ be a Lipschitz constant for g, and let $(p_\gamma)_{\gamma\in\Gamma}$ be an (o)-net, for which the inequality (3) from Proposition 3.4 holds with $\phi(x,h) = \frac{\Delta^2\Psi(x,h)}{h^2}$ (taking into account that Ψ is a major function of order 2 for f). Now to use Lemma 8.4 we put for every $\gamma \in \Gamma$ and $x \in]a,b[$ with $0 < h \le \gamma(x)$:

$$v = \underline{D}^2\Psi(x)\,, \qquad y = g(x)\,, \qquad p = p_\gamma\,, \qquad w = w_\gamma\,,$$
$$u = \frac{\Delta^2\Psi(x,h)}{h^2}\,, \qquad z = g(x-h)\,.$$

Then we obtain from (11)

$$\frac{\Delta^2\Psi(x,h)}{h^2} g(x-h) \ge \underline{D}^2\Psi(x)\,g(x) - \left[\underline{D}^2\Psi(x)\right]^+ w_\gamma - p_\gamma g(x)\,.$$

Summing up all the above estimations including (14), (15), (17), (18), (19), (20) and taking into account positivity and monotonicity of g we eventually get for all $\gamma \in \Gamma$ and $x \in]a,b[$:

$$\begin{aligned}\inf_{0<h\le\gamma(x)} \frac{\Delta^2 S(x,h)}{h^2} \ge & -Lz_\gamma - 2Cz_\gamma + \underline{D}^2\Psi(x)\,g(x) \\ & - \left[\underline{D}^2\Psi(x)\right]^+ w_\gamma - p_\gamma g(x) \\ & + \Psi'(x)g'(x) - \frac{1}{3}\gamma(x)\,L^*\,M^* - K_0\,z_\gamma.\end{aligned}$$

Hence by Definition 3.2 of the global lim inf we obtain the inequality

$$\underline{D}^2 S(x) \ge \underline{D}^2\Psi(x)\,g(x) + \Psi'(x)g'(x) \quad \text{for all} \quad \mathrm{x} \in]\mathrm{a},\mathrm{b}[\,.$$

As Ψ is a major function of order 2 for f and $g(x) \ge 0$ for every $x \in [a,b]$, then we get the existence of a set $E \subset]a,b[$ such that $]a,b[\setminus E$ is countable and

$$\underline{D}^2 S(x) \ge f(x)\,g(x) + \Psi'(x)g'(x) \quad \text{for each} \quad x \in E\,.$$

Since $g'(x) \ge 0$, from Theorem 7.9 we finally obtain:

$$\underline{D}^2 S(x) \ge f(x)\,g(x) + {I_2}'(x)\,g'(x) \quad \text{for any} \quad x \in E\,.$$

So, we have proved that S is a major function for $f \cdot g + {I_2}' \cdot g'$. Analogously, if we define

$$Z(x) = \Phi(x)g(x) - (\mathcal{P}^1)\int_a^x \Phi \cdot g', \quad x \in [a,b]\,, \tag{21}$$

where Φ is *any* minor function of order 2 for f, we prove that Z is a minor function of order 2 for $f \cdot g + I_2' \cdot g'$.

Let now $\mathcal{F}$ and $\mathcal{S}$ be the classes of all major and minor functions of order 2 for $f \cdot g + I_2' \cdot g'$ respectively, and $\mathcal{F}_1$ be the class of that major functions of order 2 for $f \cdot g + I_2' \cdot g'$ of the form (12). For all $x \in [a,b]$ we have

$$\begin{aligned} S(x) - I_2(x)g(x) + (\mathcal{P}^1)\int_a^x I_2 \cdot g' &= \Psi(x)g(x) - I_2(x)g(x) \\ &\quad - (\mathcal{P}^1)\int_a^x \big[\Psi(t) - I_2(t)\big]g'(t)dt\,. \end{aligned}$$

From this it follows that

$$\begin{aligned} 0 &\le \inf_{S\in\mathcal{F}} \left(\sup_{x\in[a,b]} \left(S(x) - I_2(x)g(x) + (\mathcal{P}^1)\int_a^x I_2 \cdot g' \right)\right) \\ &\le \inf_{S\in\mathcal{F}_1} \left(\sup_{x\in[a,b]} \left(S(x) - I_2(x)g(x) + (\mathcal{P}^1)\int_a^x I_2 \cdot g' \right)\right) \\ &\le \inf_{S\in\mathcal{F}_1} \left(\sup_{x\in[a,b]} \left(\Psi(x)g(x) - I_2(x)g(x) - (\mathcal{P}^1)\int_a^x \big[\Psi(t) - I_2(t)\big]g'(t)dt \right)\right) \\ &= \inf_{\Psi\in\mathcal{G}_2} \left(\sup_{x\in[a,b]} \left(\Psi(x)g(x) - I_2(x)g(x) - (\mathcal{P}^1)\int_a^x \big[\Psi(t) - I_2(t)\big]g'(t)dt \right)\right) = 0\,, \end{aligned}$$

thanks to the properties of the integral function I_2 and boundedness of g. Analogously we get:

$$\sup_{Z\in\mathcal{S}} \left(\sup_{x\in[a,b]} \left(I_2(x)\,g(x) - (\mathcal{P}^1)\int_a^x I_2 \cdot g' - Z(x) \right)\right) = 0\,.$$

Thus we have proved that the function $f \cdot g + I_2' \cdot g'$ is $\mathcal{P}^2$-integrable, and

$$(\mathcal{P}^2)\int_a^x (f \cdot g + I_2' \cdot g') = I_2(x)\,g(x) - (\mathcal{P}^1)\int_a^x I_2 \cdot g', \quad x \in [a,b]\,.$$

Now, by Theorem 7.9, I_2' is $\mathcal{P}^1$-integrable and, by virtue of the theorem of integration by parts for the $\mathcal{P}^1$-integral, we get that $I_2' \cdot g'$ is $\mathcal{P}^1$-integrable too. Thus, by Theorem 7.8, $I_2' \cdot g'$ is $\mathcal{P}^2$-integrable and

$$(\mathcal{P}^2)\int_a^b I_2' \cdot g' = (\mathcal{P}^1)\int_a^b \left[(\mathcal{P}^1)\int_a^x I_2'(t)\,g'(t)\,dt \right] dx\,.$$

Thus, by Proposition 7.2, $f \cdot g$ is $\mathcal{P}^2$-integrable too, and we get:

$$(\mathcal{P}^2)\int_a^b f \cdot g + (\mathcal{P}^1)\int_a^b \left[(\mathcal{P}^1)\int_a^x I_2'(t)\, g'(t)\, dt\right] dx = I_2(b)g(b) - (\mathcal{P}^1)\int_a^b I_2 \cdot g' .$$

$\square$

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