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Some Inequalities in Classical Spaces with Mixed Norms ⋆

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Abstract. We consider some inequalities in such classical Banach Function Spaces as Lorentz, Marcinkiewicz, and Orlicz spaces. Our aim is to explore connections between the norm of a function of two variables on the product space and the mixed norm of the same function, where mixed norm is calculated in function spaces on coordinate spaces, first in one variable, then in the other. This issue is motivated by various problems of functional analysis and theory of functions. We will currently mention just geometry of spaces of vector-valued functions and embedding theorems for Sobolev and Besov spaces generated by metrics which differ from L^p. Our main results are actually counterexamples for Lorentz spaces versus the natural intuition that arises from the easier case of Orlicz spaces (Section 2). In the Appendix we give a proof for the Kolmogorov–Nagumo theorem on change of order of mixed norm calculation in its most general form. This result shows that L^p is the only space where it is possible to change this order.

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0. Introduction

The main goal of this paper is to explore the validity of two (L^∞-MNP) and (L^1-MNP) inequalities given below. Let (T_1, Σ_1, μ_1) and (T_2, Σ_2, μ_2) be two non-atomic measure spaces with $\mu_1(T_1) = \mu_2(T_2) = 1$, and (T, Σ, μ) be their product. We take a Banach Function Space E on (T_2, Σ_2, μ_2) (the space of 'one' variable) and consider its 'analogue' $\widetilde{E}$ on T (the space of 'two' variables). Actually we

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consider the cases when E (and, hence, $\widetilde{E}$) is an Orlicz, Lorentz or Marcinkiewicz space (on [0, 1] or $[0, 1] \times [0, 1]$ respectively, with the Lebesgue measure).

We are interested in two following conditions on E, which we call L^∞-*Mixed Norm Property* (L^∞-MNP) and L^1-*Mixed Norm Property* (L^1-MNP) respectively:

(L^∞-MNP) There exists a constant $C > 0$ such that for every $K \in \widetilde{E}$ we have

$$\|K(s,t)\|_{\widetilde{E},(s,t)} \leq C \operatorname{ess\,sup}_{t \in T_1} \|K(s,t)\|_{E,s};$$

(L^1-MNP)**p** There exists a constant $C > 0$ such that for every $K \in \widetilde{E}$ we have

$$\int_{T_1} \|K(s,t)\|_{E,s} d\mu_1(t) \leq C\ \|K(s,t)\|_{\widetilde{E},(s,t)}.$$

We have been motivated by many applications of the idea of splitting a multidimensional variable. The most favorable situation appears in the L^p case where, due to the Fubini theorem and the trivial identity $\frac{1}{p} \cdot p = 1$, the change of the order of mixed norm calculation is possible:

$$L^p(T_1 \times T_2) = L^p(T_1, L^p(T_2)) = L^p(T_2, L^p(T_1)). \tag{0.1}$$

Unfortunately, if we seek something similar for more general norms we face the fact that, due to the generalized Kolmogorov–Nagumo theorem, equality (0.1) is characteristic for L^p (Appendix).

Because of the 'maximal' nature of L^1 and the 'minimal' nature of L^∞ in the class of Banach Function Spaces the Mixed Norm Properties above form a natural set of weaker hypotheses on relations between the norm in a space on the product and those on the coordinates. They proved to be useful in some applications.

In Section 2 we easily prove that both (L^∞-MNP) and (L^1-MNP) hold for Orlicz spaces (this result is hardly new). Also (L^1-MNP) holds for Lorentz spaces and (L^∞-MNP) holds for Marcinkiewicz spaces. It is more interesting and much less trivial that usually the other property is false. This imposes restrictions on the use of the corresponding inequalities, which is reasonable to take into account in some applications. Theorems 2.6 and 2.10, our main results, are counterexamples.

Let us discuss some possible implications. First of all, since our main results are counterexamples they can provide a researcher with the idea of a more complicated answer or of use the tools more advanced than expected.

In the L^p case it is well known that the space of traces for a Sobolev space is a suitable Besov space. The proof is based on splitting a multidimensional variable and on formula (0.1), which is no longer true if we consider a Sobolev space generated by any metric more general than L^p-metrics. So, in this case it is necessary to adopt quite a different technique, and the resulting trace space is described in terms different from those describing a Besov space (see [8, 10]). The same idea applies to estimation of singular integral operators in the Banach-space-valued setting (see [7]).

Nevertheless, even our easier results on validity of (L^1-MNP) for Lorentz spaces have been generalized to the non-commutative setting with applications to the existence of certain bases in the non-commutative rearrangement invariant spaces of operators (see [36]).

Another source of motivation was found in the literature on the geometry of spaces of vector-valued functions. Several theorems of the form:

> if X is a Banach space with the property ($\mathcal{P}$), then the Banach space $L^p(X)$ possesses the property ($\mathcal{P}$),

have been established for various properties ($\mathcal{P}$). We quote here: [37] where ($\mathcal{P}$) is the Radon-Nikodym Property [14] (Theorem 5.1) where ($\mathcal{P}$) is the property of existence of an isomorphic copy of c_0, and [32] (Theorem 1) where ($\mathcal{P}$) is the property of existence of an isomorphic copy of l^1. All these results employ formula (0.1). This was a serious restriction for generalizing such proofs to the spaces $E(X)$ where E is a general Banach Function Space. It was possible to observe that the proofs in [14, 37] actually relied on a weaker fact, which is exactly (L^∞-MNP) and (L^1-MNP) above. Hence, our counterexamples show that for Lorentz and Marcinkiewicz spaces (including separable and reflexive) we cannot carry out such proofs, though it is possible to prove the corresponding geometric facts by means of other techniques. After the introduction of semi-embedding techniques this issue has lost partly its importance (see, e.g., [13]).

It is impossible to cover here all the rich history of the inequalities considered in this paper. To mention just a few sources we refer to the series of publications of M. Milman (see, e.g., [27, 28]), devoted to the same and more general classes of spaces, and to the papers of Bardaro, Musielak and G. Vinti on the inequalities for modular function spaces (see [2]). Inequalities (L^1-MNP) and (L^∞-MNP) were introduced by Bukhvalov in [6], where several results were announced. After the presentation at the Seventh Meeting on Real Analysis and Measure Theory (July 1996; Ischia, Italy) the authors finished the main body of this joint paper by 1997 reporting then about it at a number of conferences and seminars including the International Meeting ‘Positivity and Its Applications’ (June 1998; METU, Ankara, Turkey). Reference [36], devoted mainly to non-commutative setting, and inspired by [6] and presentation at the Meeting on Real Analysis (see, [36], p.278), has some minor overlapping with the results in Subsection 2.4 (for example, Proposition 4.2 of [36] is a very special case of Theorem 2.10).

1. Preliminaries

Generally, we follow the terminology and notation of [15] and [18]. For convenience of the readers we remind here several most important definitions and results.

1.1. BANACH FUNCTION SPACES

DEFINITION 1.1 Let (T, Σ, μ) be a measure space. Denote by $L^0 = L^0(T, \Sigma, \mu)$ the space of all measurable a.e. finite functions. An *ideal space* on (T, Σ, μ) is a linear subset E of the space L^0 such that

$$\{x \in L^0,\ y \in E,\ |x| \leq |y|\} \Longrightarrow \{x \in E\}.$$

A norm $\|\cdot\|$ on an ideal space E is called *monotone* if

$$\{x, y \in E, |x| \leq |y|\} \Longrightarrow \{\|x\| \leq \|y\|\}.$$

A *Banach Function Space* (BFS for short) is an ideal space E endowed with a monotone norm with respect to which E is a Banach space.

For any BFS E we define the *Banach function dual* E' by

$$E' = \left\{x' \in L^0 : \int_T |xx'|\, d\mu < \infty \quad \forall x \in E\right\}.$$

The dual space E' can be identified with the space of integral functionals on E and hence with the space of order continuous functionals.

Many properties of a BFS E can be expressed using the following conditions:

(A) if $(x_n)_n$ is a sequence in E such that $x_n \downarrow 0$ then $\|x_n\| \to 0$;

(B) if $0 \leq x_n \uparrow,\ x_n \in E$ for every $n \in \mathbb{N}$ and $\sup_n \|x_n\| < \infty$ then there exists $x \in E$ such that $x_n \uparrow x$;

(C) if $x_n \uparrow x \in E$ then $\|x_n\| \to \|x\|$.

Note, that in the literature property **(C)** is called also weak Fatou property.

1.2. REARRANGEMENT INVARIANT SPACES

DEFINITION 1.2 For every $x \in E$ we can introduce its distribution function $\mu_x(t)$ defined by the formula:

$$\mu_x(t) = \mu\{\tau \in T : |x(\tau)| > t\};$$

two functions $x, y \in E$ are said to be *equimeasurable* ($x \sim y$) if

$$\mu_x(t) = \mu_y(t) \qquad \textit{for every } t \in \mathbb{R}.$$

DEFINITION 1.3 A BFS E on (T, Σ, μ) is a *rearrangement invariant space* (RIS for short) if $x \in E$ and $x \sim y$ imply that

$$y \in E \text{ and } \|x\| = \|y\|.$$

DEFINITION 1.4 Given a measurable function $x \in L^0(T, \Sigma, \mu)$, we call *non-increasing rearrangement* of $|x|$ the function $x^* : T \to \mathbb{R}$, defined by the formula

$$x^*(t) \equiv \inf\{\alpha \geq 0 : \mu(\tau \in T : |x(\tau)| > \alpha) \leq t\}.$$

If (T, Σ, μ) is the Lebesgue measure on $[0, 1]$ or $[0, \infty)$ then

$$\{x \in E \iff x^* \in E\} \iff \{E \text{ is a RIS}\}.$$

Now we remind the definitions of some important families of rearrangement invariant spaces, i.e. Orlicz spaces L_M, Lorentz spaces $\Lambda(\psi, p)$ and Marcinkiewicz spaces $M(\psi, p)$.

DEFINITION 1.5 Let $M : [0, \infty) \to [0, \infty)$ be a convex and increasing function, such that

$$M(0) = 0 \text{ and } M(u) \to \infty \text{ as } u \to \infty.$$

The space of all measurable functions x such that there exists a positive real number $\lambda = \lambda(x)$ for which

$$I_M(x, \lambda) \equiv \int_T M\left(\frac{|x(t)|}{\lambda}\right) d\mu < \infty, \tag{1.2}$$

with the Luxemburg norm $\|x\|_M \equiv \inf\{\lambda > 0 : I_M(x, \lambda) \leq 1\}$ is called the *Orlicz space* L_M.

DEFINITION 1.6 (see [19, 34]) Let $1 \leq p < \infty$, and $\psi : [0, \infty) \to [0, \infty)$ be an increasing concave function such that

$$\psi(0) = 0, \lim_{t\to\infty} \psi(t) = \infty \quad \text{and} \quad \lim_{t\to 0} \frac{t}{\psi(t)} = 0.$$

The space of all measurable functions x such that

$$\begin{aligned}\|x\|_{\Lambda(\psi,p)} &\equiv \left(\int_0^\infty \psi(\mu\{s : |x(s)|^p > \tau\}) d\tau\right)^{1/p} = \\ &= \left(\int_0^\infty (x^*(t))^p \, d\psi(t)\right)^{1/p} < \infty\end{aligned}$$

is called the *Lorentz space* associated to ψ and p and it is denoted by $\Lambda(\psi, p)$, or simply $\Lambda(\psi)$ when $p = 1$.

Analogously, the space of all measurable functions x such that

$$\|x\|_{M(\psi,p)} \equiv \sup_{0<h<\infty} \frac{1}{\psi(h)} \left(\int_0^h (x^*(t))^p \, dt\right)^{1/p} < \infty,$$

is called the *Marcinkiewicz space* associated to ψ and p and it is denoted by $M(\psi, p)$ or $M(\psi)$ when $p = 1$.

It is well known that $\Lambda(\psi)' = M(\psi)$ and $M(\psi)' = \Lambda(\psi)$ (see [15] and, for general p, see [34]).

DEFINITION 1.7 Let (T, Σ, μ) be a measure space. We denote by the symbol $L(p, q)$ the space of all measurable functions $x : T \to \mathbb{R}$ such that $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \begin{cases} \left[\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p}x^*(t))^q \cdot t^{-1}\,\mathrm{d}t\right]^{1/q}, & \text{if } 1 \le p < \infty,\ 1 \le q < \infty \\ \sup\limits_{t>0} t^{1/p}x^*(t), & \text{if } 1 \le p \le \infty,\ q = \infty. \end{cases}$$

Let us state several known and elementary relations among the three spaces mentioned above.

PROPOSITION 1.8 *For all* $1 \le q < p < \infty$ *and* $0 < \alpha < 1$ *we have:*
(a) $\Lambda(t^{q/p}, q) = L(p, q)$;
(b) $\Lambda(\alpha) = L(\frac{1}{\alpha}, 1)$;
(c) $M(\alpha) = L(\frac{1}{1-\alpha}, \infty)$.

1.3. SPACES WITH MIXED NORM

We need to establish a uniform system of notation for product measure spaces and associated BFS. Let (T_1, Σ_1, μ_1) and (T_2, Σ_2, μ_2) be two σ-finite measure spaces, and (T, Σ, μ) be their product. We usually denote by E a BFS on T_2, with s as a typical notation for a variable and x as a notation for a function. We consider various spaces on T_1 (e.g., L^1, L^∞, F) with t as a typical notation for a variable. Finally, we denote by $K(s, t)$ a measurable function on the product. To emphasize the role of the corresponding variables we use the self-explanatory notations like $\|K(s, t)\|_{E,s}$, $\|K(s, t)\|_{F,t}$, or $\|K(s, t)\|_{\widetilde{E},(s,t)}$.

DEFINITION 1.9 Let F be a BFS on T_1 and E be a BFS on T_2. Suppose that E satisfies condition **(C)**. Denote by $F[E]$ the space:

$$\begin{aligned} F[E] \;=\; \{K \in L^0(T, \Sigma, \mu) :\ & \text{the function } t \mapsto \|K(\cdot, t)\|_{E,s} \in F; \\ & \|K\|_{F[E]} \equiv \|\, \|K(s, t)\|_{E,s}\|_{F,t} < \infty\}. \end{aligned}$$

We call $F[E]$ the *space with mixed norm*.

We impose property **(C)** to ensure measurability of the function $t \mapsto \|K(\cdot, t)\|_{E,s}$ (see [24]; in [25] a couterexample for measurability is given). This is not an actual restriction since all concrete spaces possess **(C)**. In [3] one can find the proofs of the fact that $F[E]$ is a BFS and of other elementary properties.

The main aim of this work is to establish some relations between the mixed norm generated by a BFS on the coordinate spaces with the *similar space* on the product. What do we mean by *similar*? If a BFS E on T_2 is one of the classical spaces defined on any measure space then no problem arises at all. Say, $L_M(T_2)$ is in an obvious sense similar to $L_M(T)$ (though they can be even not isomorphic depending on the properties of the measure space T_1). In any such situation we denote the space on T by $\widetilde{E}$. Certainly, if, say, L_M is an Orlicz space on [0, 1], then $\widetilde{L_M}$ is just L_M in two variables. The same applies to Lorentz and Marcikiewicz spaces.

It is less obvious what to do if E is a general RIS on T_2, and we do not have any definition of E for T. It is possible to introduce a general formalism following the ideas of [11] on spaces of equimeasurable functions. We do not need to proceed this far here.

1.4. CALDERON–LOZANOVSKII CONSTRUCTION

To reduce the consideration of spaces $\Lambda(\psi, p)$ and $M(\psi, p)$ to that of $\Lambda(\psi)$ and $M(\psi)$ we use the so called, *Calderon–Lozanovskii construction* (see [34] for the use of the same idea to describe the dual of $\Lambda(\psi, p)$ using that of $\Lambda(\psi)$).

DEFINITION 1.10 Given two BFS E_0 and E_1 on the same measure space and $0 < s < 1$, we denote by $E_0^s E_1^{1-s}$ the BFS of all measurable functions e such that there exist $x \in E_0$ and $y \in E_1$, with $\|x\|_{E_0} = \|y\|_{E_1} = 1$, and $\gamma \geq 0$, with $|e| \leq \gamma x^s y^{1-s}$, equipped with the norm

$$\|e\|_{E_0^s E_1^{1-s}} \equiv \inf\{\gamma : |e| \leq \gamma x^s y^{1-s},\ \|x\|_{E_0}, \|y\|_{E_1} = 1\}.$$

Many useful properties and generalizations of the construction (including important duality equalities) have been invented by Lozanovskii (see [20]–[23]). Just for the sake of completeness let us note that quite often this construction leads to an interpolation space between E_0 and E_1 (that was the reason for Calderon to introduce this construction in the beginning of 1960s). The interpolation property is actually true for all the situations in our paper. Nevertheless, we do not need it.

We use the following result about mixed norm spaces and Calderon–Lozanovskii construction (see [9]): for all BFS E_0 and E_1 when both conditions **(B)** and **(C)** hold, and for every BFS F_0 and F_1 we have

$$F_0[E_0]^{1-s} F_1[E_1]^s = F_0^{1-s} F_1^s [E_0^{1-s} E_1^s]. \tag{1.3}$$

Finally, for Lorentz spaces, we have that if p' satisfies $1/p + 1/p' = 1$, then

$$\Lambda(\psi, p) = \Lambda(\psi)^{1/p} (L^\infty)^{1/p'}. \tag{1.4}$$

2. Some Inequalities in Classical Spaces

Within this section we assume that each of the measure spaces T_1 and T_2 is [0, 1] with the Lebesgue measure. Extension to the case of continuous measure spaces with probability measures is straightforward.

2.1. EASIER CASES

We start by considering several easier cases when one or both (L^∞-MNP) and (L^1-MNP) inclusions hold. The proofs are elementary and included just for the sake of completeness.

We begin with a simple observation that, for verifying (L^∞-MNP) or (L^1-MNP), it is sufficient to check just for the set-theoretic inclusions:

$$L^\infty[E] \subset \widetilde{E} \quad \text{or} \quad \widetilde{E} \subset L^1[E].$$

Indeed, identical inclusion operator is automatically bounded since inclusion is a positive operator and the norm is monotone.

LEMMA 2.1 *Let E be a RIS from Orlicz, Lorentz or Marcinkiwiecz class. If* (L^∞-MNP) *(resp.* (L^1-MNP)*p) is true for E then* (L^1-MNP) *(resp.* (L^∞-MNP)*) is true for the BFS dual E'.*

Proof. From [4] we get

$$(L^\infty[E])' = L^1[E'], \; (L^1[E])' = L^\infty[E'].$$

This proves the assertion. □

PROPOSITION 2.2 *For the Orlicz space L_M both conditions* (L^∞-MNP) *and* (L^1-MNP) *hold.*

Proof. Let us prove, say, that (L^∞-MNP) holds. Let $K(s,t)$ be a function such that, for a.e. $t \in T_1$,

$$\int_{T_2} M\left(\frac{|K(s,t)|}{\lambda}\right) \mathrm{d}\mu_2(s) \leq 1,$$

i.e. $K \in L^\infty(T_1)[L_M]$. Then, integrating in t, we get

$$\int_{T_1}\int_{T_2} M\left(\frac{|K(s,t)|}{\lambda}\right) d\mu_1(t)\mathrm{d}\mu_2(s) \leq 1.$$

Hence, $\|K\|_{L_M(T)} \leq \lambda$. □

PROPOSITION 2.3 *For the Marcinkiewicz space $M(\psi, p)$ condition* (L^∞-MNP) *holds.*

Proof. Let A be a Σ-measurable set. As usual we set $A_t = \{s \in T_2 \colon (s,t) \in A\}$. Then, due to the Fubini-Tonelli theorem, we get

$$\mu(A) = \int_{T_1} \mu_2(A_t)\mathrm{d}\mu_1(t). \tag{2.5}$$

Let $K \in L^\infty(T_1)[M(\psi, p)]$, i.e.

$$\sup\left\{\left(\frac{\int_B |K(s,t)|^p \mathrm{d}\mu_2(s)}{\psi(\mu_2(B))}\right)^{1/p} : \quad \mu_2(B) > 0\right\} \le C \tag{2.6}$$

for a.e. $t \in T_1$. If $\mu(A) > 0$ we have

$$\left(\int_A |K(s,t)|^p \mathrm{d}\mu(s,t)\right)^{1/p} = \left(\int_{T_1}\left(\int_A |K(s,t)|^p \mathrm{d}\mu_2(s)\right)\mathrm{d}\mu_1(t)\right)^{1/p} =$$

$$= \left(\int_{T_1}\left(\frac{\int_{A_t} |K(s,t)|^p \mathrm{d}\mu_2(s)}{\psi(\mu_2(A_t))}\right)\psi(\mu_2(A_t))\mathrm{d}\mu_1(t)\right)^{1/p}$$

and, using (2.6), the concavity of ψ, formula (2.5), and Jensen inequality, we obtain:

$$\left(\int_A |K(s,t)|^p \mathrm{d}\mu(s,t)\right)^{1/p} \le \left(\int_{T_1} C^p \psi(\mu_2(A_t))\mathrm{d}\mu_1(t)\right)^{1/p} =$$

$$= C\left(\int_{T_1} \psi(\mu_2(A_t))\mathrm{d}\mu_1(t)\right)^{1/p} \le C\left[\psi\left(\int_{T_1} \mu_2(A_t)\mathrm{d}\mu_1(t)\right)\right]^{1/p} =$$

$$= C\psi(\mu(A))^{1/p}.$$

This proves that the norm of K in $M(\psi, p)$ on T is less or equal C. □

COROLLARY 2.4 *For the Lorentz space* $\Lambda(\psi)$ *condition* (L^1-MNP) *holds.*

Proof. By Lemma 2.1 and Proposition 2.3 the space $E = M(\psi)$ is as required. □

Now we would like to extend this result to the general case, when $p > 1$. In the case of Lorentz spaces $\Lambda(\psi, p)$ it is not possible to change the order of integration. Though it is not the only possibility, we would like to show how the techniques from subsection 1.4 work.

PROPOSITION 2.5 *For the Lorentz space* $\Lambda(\psi, p)$, $1 \le p < \infty$, *condition* (L^1-MNP) *holds.*

Proof. From formulas (1.3) and (1.4), and Corollary 2.4 we derive:

$$\begin{aligned}\widetilde{\Lambda(\psi, p)} &= \Lambda(\psi, p)(T_1 \times T_2) = (\Lambda(\psi)(T_1 \times T_2))^{1/p}(L^\infty(T_1 \times T_2))^{1/p'} \\ &\subset (L^1(T_1)[\Lambda(\psi)])^{1/p}(L^\infty(T_1)[L^\infty(T_2)])^{1/p'} \\ &= (L^1(T_1))^{1/p}(L^\infty(T_1))^{1/p'}([\Lambda(\psi)(T_2)]^{1/p}(L^\infty(T_2))^{1/p'}) \\ &= L^p(T_1)[\Lambda(\psi, p)(T_2)] \subset L^1(T_1)[\Lambda(\psi, p)(T_2)]. \qquad \square\end{aligned}$$

2.2. LORENTZ CASE COUNTEREXAMPLES

Now we are ready to present our main result, where we show that, for Lorentz spaces, inclusion (L^∞-MNP) usually does not hold. We split our exposition in two parts: in Subsection 2.3 we consider Lorentz spaces $L(p, q)$ generated by the power function ψ. In this case we prove that for all $p > q$ condition (L^∞-MNP) is violated. Since $p \geq q$ that exhausts all possibilities.

Turning to the case of general functions ψ we start from the observation that for some pathological pairs of a function ψ and a Young function M, we have that the Lorentz space $\Lambda(\psi, p)$ coincides with the Orlicz spaces L_M (see, e.g., [33]). In this case both inclusions, (L^1-MNP) and (L^∞-MNP), hold. So we have to impose certain suitable conditions on ψ in order to construct counterexamples like in the power case. This is done in Subsection 2.4. Those sufficient conditions are quite general. Nevertheless, we do not know whether they are also necessary.

2.3. POWER CASE

THEOREM 2.6 *In the Lorentz space* $\Lambda(t^\alpha, p)$ $(0 < \alpha < 1,\ 1 \leq p < \infty)$ (L^∞-MNP) *does not hold.*

Proof. Since the embedding (L^∞-MNP) is automatically continuous then it suffices to construct a sequence $K_n \in L^\infty(T_1)[\Lambda(t^\alpha, p)]$ such that

$$\|K_n(\cdot, t)\|_{\Lambda(t^\alpha,p)} \leq 1, \qquad \forall\, n \in \mathbb{N}, \forall\, t \in T_1 \tag{2.7}$$

and

$$\lim_{n\to\infty} \|K_n\|_{\widetilde{\Lambda(t^\alpha,p)}} = \infty. \tag{2.8}$$

Fix $n \in \mathbb{N}$. Let k be an integer that will be chosen later. It can actually depend on n (as it is the case in Theorem 2.10, but we will see that it is not necessary here). Let $\{B_i^{(n)}\}_{i=1}^n$ be a partition of T_1 such that $\mu_1(B_i^n) = \frac{1}{n}$, $i = 1, \ldots, n$, and $A_i^{(n)} = \left[0, \frac{1}{k^i}\right]$ be a subset of T_2 we define

$$K_n(s, t) = \sum_{i=1}^n k^{i\alpha/p} \chi_{B_i^{(n)}}(t)\chi_{A_i^{(n)}}(s).$$

For any $h > 0$ we have $\|h^{\alpha/p}\chi_{\left[0,\frac{1}{h}\right]}\|_{\Lambda(t^\alpha,p)} = 1$. Hence, $\|K_n(\cdot,t)\|_{\Lambda(t^\alpha,p)} = 1$ for all t. That proves (2.7).

To get (2.8) we need to estimate $\|K_n\|_{\widetilde{\Lambda(t^\alpha,p)}}$ from below. By direct calculation of the norm we get:

$$\begin{aligned}\|K_n\|_{\widetilde{\Lambda(t^\alpha,p)}} &= \left\{\int_0^\infty \left(\mu\{(s,t) : K_n^p(s,t) > \tau\}\right)^\alpha \mathrm{d}\tau\right\}^{1/p} \geq \\ &\geq \left(\frac{1}{n^\alpha}\sum_{i=2}^n \left(k^{i\alpha} - k^{(i-1)\alpha}\right)\left(\sum_{j=i}^n \frac{1}{k^j}\right)^\alpha\right)^{1/p}.\end{aligned}$$

Estimating the i-th term in the expression above we get:

$$\left(k^{i\alpha} - k^{(i-1)\alpha}\right)\left(\sum_{j=i}^n \frac{1}{k^j}\right)^\alpha \geq \left(k^{i\alpha} - k^{(i-1)\alpha}\right)\frac{1}{k^{i\alpha}} = (1 - k^{-\alpha}).$$

If we choose, for instance, $k = 2$ we have that

$$\lim_{n\to\infty} \|K_n\|_{\widetilde{\Lambda(t^\alpha,p)}} \geq \lim_{n\to\infty}\left(\frac{1}{n^\alpha}(n-1)(1-2^{-\alpha})\right)^{1/p} = +\infty,$$

which implies (2.8). □

As a consequence of Theorem 2.6 and Proposition 1.8 we obtain the following results:

PROPOSITION 2.7 *For the Lorentz space $L(p,q)$ the following relations hold:*

$$\begin{aligned}&L^\infty(T_1)[L(p,q)] \not\subset \widetilde{L(p,q)} \subset L^q(T_1)[L(p,q)],\ q < p;\\ &L^\infty(T_1)[L(p,q)] \subset L^q(T_1)[L(p,q)] \subset \widetilde{L(p,q)} \not\subset L^1(T_1)[L(p,q)],\ q > p;\\ &L^\infty(T_1)[L(p,p)] \subset L^p(T_1)[L^p] \subset \widetilde{L(p,p)} = L^p = L^p(T_1)[L^p] \subset\\ &\subset L^q(T_1)[L(p,p)].\end{aligned}$$

Proof. It follows from Theorem 2.6, Lemma 2.1 and Theorem 2.5. □

COROLLARY 2.8 *For the Lorentz space $L(p,q)$ condition* (L^∞-MNP) *holds if and only if $p = q$.*

Proof. This is a consequence of Proposition 2.7. □

COROLLARY 2.9 *There exists a reflexive RIS E such that E fails to have* (L^∞-MNP), *and E' fails to have* (L^1-MNP)*p.*

Proof. Due to Lemma 2.1 and Theorem 2.6 we can pose $E = \Lambda(t^\alpha, p)$, $1 < p < \infty$. □

From Lemma 2.1 and Proposition 2.6 we see that the classical Marcinkiewicz space $M(\alpha)$ fails to have (L^1-MNP) (though (L^∞-MNP) holds).

2.4. GENERAL CASE

Now we want to expand our result to general $\Lambda(\psi, p)$. The proof of Theorem 2.6 uses intensively multiplicativity property of the generating function $\psi(t) = t^\alpha$. Thus a suitable modification of our approach is needed. As it was mentioned before, it is impossible to obtain the result given in Theorem 2.6 for every ψ. So we have to impose the following two conditions on ψ:

$$(\psi_1)] \quad \overline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)} = q < 2;$$

$$(\psi_2)] \quad \underline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)} = r > 1.$$

Both conditions hold for $\psi(t) = t^\alpha$ because

$$\lim_{t\to 0}\frac{\psi(2t)}{\psi(t)} = 2^\alpha$$

and $1 < 2^\alpha < 2$. So conditions (ψ_1) and (ψ_2) form a natural generalization of the power case.

THEOREM 2.10 *If both conditions (ψ_1) and (ψ_2) hold then the Lorentz space $\Lambda(\psi, p)$ fails to have (L^∞-MNP).*

Proof. We will construct a sequence $\{K_n\}$, again satisfying (2.7) and (2.8), now with ψ in place of t^α.

Fix $n \in \mathbb{N}$. Let k be an integer that will be chosen later. In our construction we will adjust k accordingly with n. So, formally speaking, $k = k(n)$. Nevertheless, we prefer to omit n in the notation since there is no interdependence between $k(n)$ defined for different n. Let $\{B_i^{(n)}\}_{i=1}^{2^n}$ be a partition of T_1 such that $\mu_1(B_i^{(n)}) = \frac{1}{2^n}$, $i = 1, \dots, 2^n$, and $A_i^{(n)} = [0, \frac{1}{k_n^i}]$ be a subset of T_2.
We define

$$K_n(s,t) = \sum_{i=1}^{2^n}\left(\psi\left(\frac{1}{k^i}\right)\right)^{-1/p}\chi_{B_i^{(n)}}(t)\chi_{A_i^{(n)}}(s).$$

As before we obviously have $\|K_n(\cdot,t)\|_{\Lambda(\psi,p)} = 1$ for every $t \in T_1$, hence, (2.7) is true. So, we need to proceed with the estimate, which is an analogue of (2.8).

We cannot use multiplicativity of ψ now, so our approach should be a bit more sophisticated. We have as before:

$$\|K_n\|_{\widetilde{\Lambda(\psi, p)}} = \left(\int_0^\infty \psi(\mu\{(s,t) : K_n^p(s,t) > \tau\})\mathrm{d}\tau\right)^{1/p} \geq$$

$$\geq \left\{\sum_{i=2}^{2^n}\left[\left(\psi\left(\frac{1}{k^i}\right)\right)^{-1} - \left(\psi\left(\frac{1}{k^{i-1}}\right)\right)^{-1}\right]\psi\left(\frac{1}{2^n}\sum_{j=i}^{2^n}\frac{1}{k^j}\right)\right\}^{1/p}. \quad (2.9)$$

From (ψ_1) and (ψ_2) we obtain

(ψ_1') $\quad \overline{\lim}_{t\to 0}\dfrac{\psi(2^n t)}{\psi(t)} \leq q^n$ for every $n \in \mathbb{N}$;

(ψ_2') $\quad \underline{\lim}_{t\to 0}\dfrac{\psi(2^n t)}{\psi(t)} \geq r^n$ for every $n \in \mathbb{N}$.

Let σ, ρ be arbitrarily fixed (but independent of n) numbers such that $1 < \rho < r < q < \sigma < 2$. Using (ψ_1') and (ψ_2'), we get

$$\overline{\lim}_{t\to 0}\frac{\psi(2^n t)}{\psi(t)} < \sigma^n, \qquad \underline{\lim}_{t\to 0}\frac{\psi(2^n t)}{\psi(t)} > \rho^n.$$

So, there exists $\delta = \delta(n)$ such that for every t with $0 < t \leq \delta$ we have

$$\sigma^{-n} < \frac{\psi(t)}{\psi(2^n t)} < \rho^{-n}. \quad (2.10)$$

It is the right time to choose $k = k(n)$. We impose the following two conditions:

(a) $1/k < \delta$;

(b) $k > 2^n$.

Now we want to estimate the i-th term in formula (2.9):

$$\left[\left(\psi\left(\frac{1}{k^i}\right)\right)^{-1} - \left(\psi\left(\frac{1}{k^{i-1}}\right)\right)^{-1}\right]\psi\left(\frac{1}{2^n}\sum_{j=i}^{2^n}\frac{1}{k^j}\right) \geq$$

$$\geq \left[\left(\psi\left(\frac{1}{k^i}\right)\right)^{-1} - \left(\psi\left(\frac{1}{k^{i-1}}\right)\right)^{-1}\right]\psi\left(\frac{1}{2^n k^i}\right) =$$

$$= \left[\left(\psi\left(\frac{1}{k^i}\right)\right)^{-1} - \left(\psi\left(\frac{1}{k^{i-1}}\right)\right)^{-1}\right]\frac{\psi\left(\frac{1}{2^n k^i}\right)}{\psi\left(\frac{1}{k^i}\right)} \cdot \psi\left(\frac{1}{k^i}\right) =$$

$$= \left[1 - \frac{\psi\left(\frac{1}{k^i}\right)}{\psi\left(\frac{1}{k^{i-1}}\right)}\right]\frac{\psi\left(\frac{1}{2^n k^i}\right)}{\psi\left(\frac{1}{k^i}\right)}.$$

From (b) we deduce that

$$\psi\left(\frac{1}{k^i}\right) = \psi\left(\frac{1}{kk^{i-1}}\right) \leq \psi\left(\frac{1}{2^n k^{i-1}}\right).$$

Since $\frac{1}{k^i} < \frac{1}{k} < \delta$, we have by (a) and (2.10):

$$\left[1 - \frac{\psi\left(\frac{1}{k^i}\right)}{\psi\left(\frac{1}{k^{i-1}}\right)}\right] \frac{\psi\left(\frac{1}{2^n k^i}\right)}{\psi\left(\frac{1}{k^i}\right)} \geq \left(1 - \frac{\psi\left(\frac{1}{2^n k^{i-1}}\right)}{\psi\left(\frac{1}{k^{i-1}}\right)}\right)\sigma^{-n} \geq$$

$$\geq (1 - \rho^{-n})\sigma^{-n}.$$

Taking into account that $2/\sigma > 1$ and $\rho > 1$, and passing to the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \|K_n\|_{\widetilde{\Lambda(\psi, p)}} \geq$$

$$\geq \lim_{n\to\infty}\left(\sum_{i=2}^{2^n}\left[\left(\psi\left(\frac{1}{k^i}\right)\right)^{-1} - \left(\psi\left(\frac{1}{k^{i-1}}\right)\right)^{-1}\right]\psi\left(\frac{1}{2^n}\sum_{j=i}^{2^n}\frac{1}{k^j}\right)\right)^{1/p} \geq$$

$$\geq \lim_{n\to\infty}\left((2^n - 1)\sigma^{-n}[1 - \rho^{-n}]\right)^{1/p} = \infty. \qquad \square$$

3. Appendix. Generalized Kolmogorov–Nagumo Theorem

Let F be a BFS on (T_1, Σ_1, μ_1) and E be a BFS on (T_2, Σ_2, μ_2). For a measurable function $K(s, t)$ in two variables we can construct mixed norms in the following two ways:

$$\|K\|_{t,s} = \|\|K(s, t)\|_{E,s}\|_{F,t}\,; \tag{3.11}$$

$$\|K\|_{s,t} = \|\|K(s, t)\|_{F,t}\|_{E,s}\,. \tag{3.12}$$

We assume that all operations in (3.11) and (3.12) are correctly defined. It is obviously true for the case of 'simple' functions of the form

$$K(s, t) = \sum_{i=1}^{n} x_i(s) f_i(t) \quad (x_i \in E,\, f_i \in F). \tag{3.13}$$

As it was mentioned in Introduction, if $E = L^p$ and $F = L^p$, we have

$$\|\|K(s, t)\|_{E,s}\|_{F,t} = \|\|K(s, t)\|_{F,t}\|_{E,s} \tag{3.14}$$

for any K.

Theorems of Kolmogorov–Nagumo type claim that (3.14) is true just for this trivial case. Actually from Kolmogorov [16] and Nagumo [30] about axiomatic 'averages' of functions it is not difficult to derive such a result for RIS on [0, 1] (see [29]; the term of Kolmogorov–Nagumo theorem has been introduced in this paper). In applications we mainly need not the precise (isometric) equality (3.14) but validity of a similar equivalent norms conditions, namely,

$$c_1 \| \|K(s,t)\|_{F,s} \|_{E,t} \le \| \|K(s,t)\|_{E,t} \|_{F,s} \le c_2 \| \|K(s,t)\|_{F,s} \|_{E,t}, \tag{3.15}$$

where the constants $c_1, c_2 > 0$ do not depend on K. We will see that it does not enlarge the list of spaces, which still reduces to L^p.
The techniques of Kolmogorov–Nagumo give no clue to this much more general setting. Using more modern techniques of unconditional basic sequences Nielsen ([31]) was able to deal with the case when the BFS E and F have order continuous norm (condition **(A)**). The most general form of Kolmogorov–Nagumo theorem was derived, as far as we know, by Bukhvalov in [5] (without the proof it was announced in [7, 8]), where no restrictions on E and F were imposed. Since the proof was given in the framework of Banach lattices rather than BFS, it made difficult to follow the proofs for those not in the area. So, it became obvious that it is desirable to provide a reader with the modified proof, which directly works for BFS. Here it is.

For a weight function $w > 0$ on T_1 denote by $L^p(w\mathrm{d}\mu_1)$ the weighted L^p-space $(1 \le p \le \infty)$ with the norm

$$\|x\|_{p,w} = \begin{cases} \left(\int |x(t)|^p w(t) \mathrm{d}\mu_1(t) \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t \in T_1} |x(t)| w(t), & p = \infty. \end{cases}$$

Let us recall that a BFS E is an AM-space provided $\| |x_1| \vee |x_2| \| = \max(\|x_1\|, \|x_2\|)$. If such space possesses properties **(C)** and **(B)** then E coincides with a weighted L^∞-space.

THEOREM 3.1 **(Generalized Kolmogorov–Nagumo theorem)** *If for the BFS E and F equivalence inequalities* (3.15) *hold for every function K of the form* (3.13)*, where the functions x_i (respectively, f_i) are pairwise disjoint, then there exists a number $p \in [1, \infty]$ such that*

(1) *if $p < \infty$, then there are weights w_1 and w_2 such that the norm of the BFS E is equivalent to the norm $\| \cdot \|_{p,w_1}$ and the norm of the BFS F is equivalent to the norm $\| \cdot \|_{p,w_2}$; hence, E and F coincide with the corresponding weighted L^p-spaces, with equivalent norms;*

(2) *if $p = \infty$, then E and F both have norms equivalent to the norms in AM-spaces; if additionally E and F both possess properties* **(C)** *and* **(B)** *then these spaces coincide with some weighted L^∞-spaces, with equivalent norms.*

REMARK 3.2 An isometric variant of Theorem 3.1 holds, i.e. if (3.14) holds for all the functions of the form (3.13), then we have equality of the norms rather than equivalence. This is an easier result, which can be derived from the proof below.

In order to prove Theorem 3.1 we need the following two results. The first result is a deep Theorem A due to Krivine [17] about finite lattice representability of a suitable l^p in an arbitrary Banach lattice. The second result is a general characterization of L^p-spaces in the class of BFS (Banach lattices).

THEOREM A (Krivine [17]). *Let E be an arbitrary infinite-dimensional Banach lattice (e.g., BFS). Then there is a number $p \in (1, \infty]$ such that for any $\varepsilon > 0$ and for any natural number n there exists a set of pairwise disjoint norm one elements $x_1, x_2, \dots, x_n \in E_+$, which for any set of scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ satisfies the following inequalities:*

$$(1-\varepsilon)\left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{n} \lambda_i x_i\right\| \leq (1+\varepsilon)\left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{1/p}. \tag{3.16}$$

When $p = \infty$ we mean usual modification.

THEOREM B. (i) *The norm in a BFS E is equivalent to a $\|\cdot\|_{p,w}$-norm for some $p \in [1, \infty)$ if and only if there exist constants $c_1, c_2 > 0$ such that*

$$c_1\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{n} x_i\right\| \leq c_2\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \tag{3.17}$$

for any pairwise disjoint $x_1, x_2, \dots, x_n \in E_+$.
(ii) *The norm in a BFS E is equivalent to the norm of an AM-space if and only if there exists a constant $c > 0$ such that*

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq c \max_{i=1..n} \|x_i\| \tag{3.18}$$

for any pairwise disjoint $x_1, x_2, \dots, x_n \in E_+$.

Assertion (i) is due to Mayer-Nieberg [26] and assertion (ii) is exactly Theorem 5 by Abramovich [1].

Proof of Theorem 3.1. Let $p \in [1, \infty]$ be a number, which corresponds to the BFS F in accordance with Theorem A. We will prove that for this p either (3.17) or (3.18) holds for E. This implies that, due to Theorem B, E is isomorphic to a weighted L^p-space.

To simplify the notations, let $1 \leq p < \infty$ (the case of $p = \infty$ is just easier). Take arbitrary pairwise disjoint elements $x_1, x_2, \dots, x_n \in E_+$. Using the assertion of Theorem A we find, for the given n, a set of pairwise disjoint elements

$f_1, f_2, \dots, f_n \in F_+$ with norm one, satisfying (3.16). Now construct $K(s,t)$ in accordance to formula (3.13) using those $\{x_i\}$ and $\{f_i\}$. The disjointness conditions imply that

$$\|K\|_{t,s} = \left\| \sum_{i=1}^{n} \right\| f_i \|_F \; x_i \|_E, \quad \|K\|_{s,t} = \left\| \sum_{i=1}^{n} \right\| x_i \|_E \; f_i \|_F.$$

Taking into account formulas (3.15) and (3.16) together with $\|f_i\| = 1$ we get

$$\left\| \sum_{i=1}^{n} x_i \right\|_E \le \frac{1}{c_1} \left\| \sum_{i=1}^{n} \right\| x_i \|_E \; f_i \|_F \le \frac{1+\varepsilon}{c_1} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p};$$

$$\left\| \sum_{i=1}^{n} x_i \right\|_E \ge \frac{1}{c_2} \left\| \sum_{i=1}^{n} \right\| x_i \|_E \; f_i \|_F \ge \frac{1-\varepsilon}{c_2} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}.$$

This implies that (3.17) is true for E and, hence, the norm $\|\cdot\|_E$ is equivalent to $\|\cdot\|_{p,w_1}$.

Verifying that (3.17) is true for F with the same p is analogous to the above with the only difference that we do not need rely on Theorem A this time. □

One of the principal consequences of Theorem 3.1 is impossibility of splitting of any RIS of many variables (different from L^p) into a mixed norm space of coordinate variables. This implies specific features of embedding and trace theorems for Sobolev and Besov spaces generated by non L^p-metrics. Such theorems were obtained in [8, 10]. Being precise we can formulate the following corollary.

COROLLARY 3.3 *Let G be a RIS on $T = T_2 \times T_1$, which is different from L^p (as a set). Assume that T_1 and T_2 are not reduced to finite number of atoms. Then there are no BFS F on T_1 and E on T_2 such that $G = F[E]$.*

Proof. The space $F[E]$, where at least one of spaces E or F are different from L^p, can not be a RIS since in accordance to Theorem 3.1 such space cannot be invariant with respect to the mapping $(s,t) \to (t,s)$, which is an automorphism of T. □

REMARK 3.4 (i) In [5] a number of related results have been derived. Among them there is a deeper investigation of the case of AM-spaces. Other generalizations and corollaries include the results on equivalence of l- and m-norms in tensor products and spaces of operators (see the definitions in [35]). Some applications to duality of operators with abstract norm (in the sense of Kantorovich, see [15]) are given as well.

(ii) In [6] Theorem 3.1 is applied to description of averages of measurable families of convex bodies in R^n (cf. [12]).

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