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A McShane integral for multifunctions

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Abstract

In this paper we introduce a "generalized" McShane integration for Banach-valued multifunctions with weakly compact and convex values and we give also a comparison between this integration and the Aumann integration.

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1. Introduction

The notion of integral of a multivalued function is very useful in many branches of mathematics like mathematical economics, control theory, differential inclusions, convex analysis, etc. It has been introduced by many authors and in different ways. The first was Aumann in 1965, in order to apply it to general equilibria in economics. This integral was built using selections, but some properties were missing, so Debreu introduced the multivalued Bochner integral. In both cases the definition of measurable multifunction is crucial since it is necessary to ensure that at least a selection exists. Many authors worked on the problem of measurability of multifunctions; we quote here for example [4, 13, 11, 9, 10, 2] for the countably additive case and [20] for a review in the finitely additive case.

Here we introduce a new kind of multivalued integral which does not need a priori the notion of measurability; this fact looks interesting for example in differential inclusions. The idea comes out from a discussion with Prof. Jan Andres during a congress in 2000 and was presented in 2003 at the XVII Congress of U.M.I..

Our starting point is a paper by Jarník and Kurzweil [14] in which the authors proposed a new definition based on Kurzweil-Henstock "selections" for $\mathbb{R}^n$-valued multifunctions, defined in a bounded interval of $\mathbb{R}$. Jarník and Kurzweil applied it to differential inclusions and showed that under suitable conditions (namely compactness of values) this integral coincides with the Aumann's one.

Here we extend these results in two directions: we consider in fact multifunctions defined in the whole real line and moreover taking values in a Banach space not necessarily separable. In particular in section 3 we introduce the $(\star)$-integral by using McShane integrable single valued functions and then we compare it with the Aumann integral. Finally, in section 4, making use of the Rådstrom embedding theorem, the McShane multivalued integral is introduced and compared with the $(\star)$ and Aumann integrals. When the McShane multivalued integral exists, then the $(\star)$-integral exists too and it coincides with it, and so all the properties of the single valued McShane integral are inherited by the multivalued one.

2. Preliminaries and known results on the generalized McShane integral.

The generalized McShane integral (McShane integral briefly), as a limit of suitable Riemann sums, was developed in the vector valued case by Fremlin in [7]. In this section, we assume that S is a space and $\mathcal{T}$ a topology on S making $(S, \mathcal{T}, \Sigma, \mu)$ a non-empty σ-finite quasi-Radon measure space which is *outer regular,* namely such that

$$\mu(B) = \inf\{\mu(G) : B \subseteq G \in \mathcal{T}\} \quad \forall\, B \in \Sigma.$$

A *generalized McShane partition* P of S ([7, Definitions 1A]) is a disjoint sequence $(E_i, t_i)_{i\in\mathbb{N}}$ of measurable sets of finite measure, with $t_i \in S$ for every $i \in \mathbb{N}$ and $\mu(S \setminus \bigcup_i E_i) = 0$.
A *gauge* on S is a function $\Delta : S \to \mathcal{T}$ such that $s \in \Delta(s)$ for every $s \in S$. A generalized McShane partition $(E_i, t_i)_i$ is *subordinate to* Δ if $E_i \subset \Delta(t_i)$ for every $i \in \mathbb{N}$.
From now on with the symbol $\mathcal{P}$ we denote the class of all generalized McShane partitions of $[a, b]$, and with $\mathcal{P}_\Delta$ those elements from $\mathcal{P}$ that are subordinate to Δ.
Let X be a Banach space. We say that:

Definition 1 [7, Definitions 1A] A function $f : S \to X$ is *McShane integrable,* with integral w, if for every $\varepsilon > 0$ there exists a gauge $\Delta : S \to \mathcal{T}$ such that

$$\limsup_{n\to+\infty} \left\| w - \sum_{i=1}^{n} \mu(E_i) f(t_i) \right\| \le \varepsilon$$

for every generalized $\mathcal{P}_\Delta$ McShane partition $(E_i, t_i)_i$. In this case, we write $\int_S f = w$.

For the properties of the McShane generalized integral we suggest the quoted article [7] by Fremlin. Here we recall only this result which will be used later:

[7, Lemma 1J] Let $f : S \to X$ be a function. Then, for every $\varepsilon > 0$, there exists a gauge $\Delta : S \to \mathcal{T}$ such that

$$\sum_{i=1}^{\infty} \mu(E_i)\, \|f(t_i)\| \leq \overline{\int}_S \|f(t)\|\, \mu(dt) + \varepsilon,$$

whenever $(E_i, t_i)_i$ is a generalized $\mathcal{P}_\Delta$ McShane partition of S and $\overline{\int}_S \|f(t)\|\, \mu(dt)$ denotes outer integration, namely

$$\overline{\int}_S \|f(t)\|\, \mu(dt) := \inf \left\{ \int_S g(t)\mu(dt), g \in L^1(\mathbb{R}), \|f(t)\| \leq g(t) \right\}.$$

Fremlin in [7] studied also the relationship among this integral and the usual "strong" and "weak" integrals in Banach spaces. In particular this new integral, which coincides with the classical one in $\mathbb{R}$, is weaker than the Bochner and stronger than the Pettis one. In fact Bochner integrability implies McShane integrability and the two integrals agree ([7, Theorem 1K]), while McShane integrability implies Pettis integrability and the two integrals agree ([7, Theorem 1Q]). Moreover, if the Banach space X is separable, then McShane and Pettis integrability coincide ([7, Corollary 4C]).

3. Applications to multivalued integration

Here we introduce a new kind of multivalued integral. There are in the literature several papers on Aumann integration and other multivalued integrations; see for example [1], [20] and their bibliography. Note that, in all existing multivalued integration theories, in order to define the multivalued integrals, a notion of measurability or "total measurability" is required. For the kind of integrability that we will introduce, no measurability is required a priori and so we can define a multivalued integral also in non separable Banach spaces.

Throughout this section, let $S = [a, b]$, where $a, b \in [-\infty, +\infty]$, $a < b$. Moreover, assume that $\mathcal{T}$, Σ and μ are respectively the families of all open subsets of $[a, b]$, the σ-algebra of all Lebesgue measurable subsets of $[a, b]$ and the Lebesgue measure on $[a, b]$ respectively.

Let $cwk(X)$ $[ck(X)]$ denote the family of all convex and weakly compact [respectively convex and compact] subsets of a Banach space X. We denote with the symbol $d(x, C)$ the usual distance between a point and a nonempty set $C \subset X$, namely $d(x, C) = \inf\{\|x - y\| : y \in C\}$, and by $\mathcal{U}(C, \varepsilon)$ the ε-neighborhood of the set C, i.e.

$$\mathcal{U}(C, \varepsilon) = \{z \in E : \exists\, x \in C \ \text{ with } \ \|x - z\| \leq \varepsilon\}.$$

Observe that, if C is convex, then $\mathcal{U}(C, \varepsilon) = \mathrm{co}(\mathcal{U}(C, \varepsilon))$.
If C, D are two nonempty subsets of X, we denote with the symbol $e(C, D)$ the excess of C with respect to D, namely $e(C, D) = \sup\{d(x, D) : x \in C\}$, while the Hausdorff distance between C and D is $h(C, D) = \max\{e(C, D), e(D, C)\}$. We remember that $h(C, D) = 0$ if and only if $cl\{C\} = cl\{D\}$, where the symbol $cl\{\cdot\}$ denotes the closure of the considered set with respect to the norm topology.

Like in [14] we define a multivalued integral in the following way:

Definition 2 Let $F : [a, b] \to 2^X \setminus \emptyset$ be a multifunction. We call $(*)$-*integral* of F over $[a, b]$ the set $\Phi(F, [a, b])$ given by:

$$\begin{aligned}\Phi(F, [a, b]) \;=\; & \{x \in X : \forall\, \varepsilon > 0, \exists \text{ a gauge } \Delta : \text{ for every generalized} \\ & \mathcal{P}_\Delta \text{ McShane partition} (E_i, t_i)_{i \in \mathbb{N}} \text{ there holds} \\ & \limsup_n \, d(x, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \varepsilon\},\end{aligned}$$

where, as usual, $\sum_{i=1}^n F(t_i)\mu(E_i) := \{\sum_{i=1}^n x_i \mu(E_i) : \ x_i \in F(t_i)\}$.

Observe that, if F is single-valued, then $\Phi(F,[a,b])$ coincides with the McShane integral, if it exists. We now show that:

Proposition 1 *If F is bounded valued, then*

$$\Phi(F,[a,b]) = \bigcap_{\varepsilon>0}\bigcup_{\Delta}\bigcap_{(E_i,t_i)_i\in\mathcal{P}_\Delta}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\mathcal{U}(\sum_{i=1}^{n}F(t_i)\mu(E_i),\varepsilon). \qquad (1)$$

Proof. Let $z\in\Phi(F,[a,b])$; for every $\varepsilon>0$, there exists a gauge $\Delta(\varepsilon/2)$ such that for every generalized $\mathcal{P}_\Delta$ McShane partition $(E_i,t_i)_i$

$$\limsup_n d(z,\sum_{i=1}^{n}F(t_i)\mu(E_i)) = \inf_{m\geq 1}\sup_{n\geq m} d(z,\sum_{i=1}^{n}F(t_i)\mu(E_i)) \leq \varepsilon/2.$$

From this it follows that there exists $m\in\mathbb{N}$ such that

$$d(z,\sum_{i=1}^{n}F(t_i)\mu(E_i)) \leq \varepsilon \quad \text{for every } n\geq m,$$

and thus

$$z\in\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\mathcal{U}(\sum_{i=1}^{n}F(t_i)\mu(E_i),\varepsilon).$$

Hence, $z\in\bigcap_{\varepsilon>0}\bigcup_{\Delta}\bigcap_{(E_i,t_i)_i\in\mathcal{P}_\Delta}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\mathcal{U}(\sum_{i=1}^{n}F(t_i)\mu(E_i),\varepsilon)$.

Conversely, let $z\in\bigcap_{\varepsilon>0}\bigcup_{\Delta}\bigcap_{(E_i,t_i)_i\in\mathcal{P}_\Delta}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\mathcal{U}(\sum_{i=1}^{n}F(t_i)\mu(E_i),\varepsilon)$. Then, for every $\varepsilon>0$, there exists a gauge Δ such that, for every generalized $\mathcal{P}_\Delta$ McShane partition $(E_i,t_i)_i$,

$$z\in\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\mathcal{U}(\sum_{i=1}^{n}F(t_i)\mu(E_i),\varepsilon),$$

which means that for every $\varepsilon>0$, there exists a gauge Δ such that, for every generalized $\mathcal{P}_\Delta$ McShane partition $(E_i,t_i)_i$,

$$\limsup_n d(z,\sum_{i=1}^{n}F(t_i)\mu(E_i)) \leq \varepsilon,$$

namely $z\in\Phi(F,[a,b])$. □

Remark 1 **(a)** Observe that, by definition, the set $\Phi(F,[a,b])$ is closed; in fact if $(z_n)_n$ is a sequence in $\Phi(F,[a,b])$ which converges to $z \in X$ then, for every $\varepsilon > 0$ there exist an integer k and a gauge Δ_k such that for every generalized $\mathcal{P}_{\Delta_k}$ McShane partition $(E_i, t_i)_i$

$$\|z - z_k\| \leq \varepsilon/2, \quad \limsup_{n\to\infty} d(z_k, \sum_{i=1}^{n} F(t_i)\mu(E_i)) \leq \varepsilon/2;$$

then

$$\begin{aligned} &\limsup_{n\to\infty} d(z, \sum_{i=1}^{n} F(t_i)\mu(E_i)) \leq \\ \leq\ &\limsup_{n\to\infty} \left(\|z - z_k\| + d(z_k, \sum_{i=1}^{n} F(t_i)\mu(E_i)) \right) \leq \varepsilon. \end{aligned}$$

and therefore, by definition, $z \in \Phi(F,[a,b])$.

(b) Moreover, if F is closed and convex valued, $\Phi(F,[a,b])$ is convex too. In fact, since

$$\Phi(F,[a,b]) = \bigcap_{\varepsilon>0} \bigcup_{\Delta} \bigcap_{(E_i,t_i)_i \in \mathcal{P}_\Delta} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon), \qquad (2)$$

if $x, y \in \Phi(F,[a,b])$ then for every $\varepsilon > 0$ there exist Δ_x, Δ_y such that

$$\begin{aligned} x &\in \bigcap_{(E_i,t_i)_i \in \mathcal{P}_{\Delta_x}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon) \\ y &\in \bigcap_{(E_i,t_i)_i \in \mathcal{P}_{\Delta_y}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon). \end{aligned}$$

Let $\Delta = \Delta_x \cap \Delta_y$. Then, for every generalized $\mathcal{P}_\Delta$ McShane partition $(E_i, t_i)_i$, we have:

$$x, y \in \bigcap_{(E_i,t_i)_i \in \mathcal{P}_{\Delta}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon)$$

and so there are two integers m_1, m_2 such that

$$x \in \bigcap_{n=m_1}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon), \quad y \in \bigcap_{n=m_2}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon).$$

If we take $m = \max\{m_1, m_2\}$ then

$$x, y \in \bigcap_{n=m}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon)$$

and so, since this last set is convex, for every $a \in [0, 1]$,

$$ax + (1-a)y \in \bigcap_{(E_i,t_i)_i \in \mathcal{P}_\Delta} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{U}(\sum_{i=1}^{n} F(t_i)\mu(E_i), \varepsilon).$$

Then the convexity of $\Phi(F, [a, b])$ follows.

(c) If F is integrably bounded, namely there exists $g \in L^1([a, b])$ such that $h(F(t), \{0\}) \leq g(t)$ a.e. , then $\Phi(F, [a, b])$ is bounded. Indeed for every $z \in \Phi(F, [a, b])$ and for every $\varepsilon > 0$ there are a gauge Δ and a point $x \in \sum_{i=1}^{n} F(t_i)\mu(E_i)$ (where $(E_i, t_i)_i$ is a generalized $\mathcal{P}_\Delta$ McShane partition) such that $\|z - x\| \leq \varepsilon$, and hence

$$\|z\| \leq \|z - x\| + \|x\| \leq \varepsilon + \|g\|_1.$$

By the arbitrariness of z, it follows that $\Phi(F, [a, b])$ is bounded.

Observe also that in the definition no separability of X, X' is required. Consider now the classical integral given in the theory of multivalued integration, namely the Aumann integral [1], which is defined by:

$$(A) - \int_a^b F dt = \left\{ (B) - \int_a^b f dt;\ f \in S_F^1 \right\},$$

where S_F^1 is the set of all Bochner integrable selections of F.
We recall also that a multifunction F is measurable if

$$F^-(C) = \{t \in [a, b] : F(t) \cap C \neq \emptyset\}$$

is a Borel set for every closed set $C \subset X$.
We want to compare now the $(*)$- and the (A)-integrals.

Proposition 2 *Let $F : [a,b] \to 2^X \setminus \emptyset$ be a multifunction. Then*

$$(A) - \int_a^b F dt \subset \Phi(F, [a,b]).$$

Proof: Since the proof is easy, we give it only for the sake of simplicity. The inclusion is obvious if the Aumann integral is empty. If it is not, let $z \in (A)\int_a^b F(t)dt$, then there exists a function $f \in S_F^1$ such that $f(t) \in F(t)$ for every $t \in [a,b]$ and $z = \int_a^b f d\mu$. Since f is Bochner integrable it is also McShane integrable and so, for every $\varepsilon > 0$, there exists a gauge Δ such that, for every $\mathcal{P}_\Delta$ generalized McShane partition $(E_i, t_i)_i$, we have

$$\limsup_n d(z, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \limsup_n \|z - \sum_{i=1}^n f(t_i)\mu(E_i)\| \leq \varepsilon$$

and thus it follows that $z \in \Phi(F, [a,b])$. $\square$.

In order to prove the opposite inclusion we suppose that the multifunction F is also $cwk(X)$-valued, measurable and integrably bounded and that the space X is separable. In this case, we will show that the (A)-integral is non empty. In order to prove this, we recall the following useful results:

Proposition 3 [12, Proposition II.5.20] Let X be a separable Banach space. *If $F : [a,b] \to cwk(X)$ is graph measurable and integrably bounded, then*

$$(A) - \int_a^b F(t)dt \in cwk(X).$$

Proposition 4 [12, Proposition II.5.2] *Let X be a separable Banach space. If $F : [a,b] \to cwk(X)$ is graph measurable and $S_F^1 \neq \emptyset$, then for every $x' \in X'$ we have:*

$$s(x', (A) - \int_a^b F(t)dt) = \int_a^b s(x', F(t))\, dt$$

where $s(x', \cdot)$ is the support function defined for any nonempty set $C \subset X$ by $s(x', C) = \sup\{< x', x >: x \in C\}$.

Lemma 1 [4, Lemma III.14] *Let* $P = (x'_n)_n$ *be a dense sequence in* X' *for the topology* $\tau(X', X)$*, and* K *be a closed, convex, weakly locally compact subset of* X *which contains no line. Then*

$$K = \cap_n \{x \in X : \ < x'_n, x > \ \leq \ s(x'_n, K)\}.$$

Moreover

Proposition 5 *Let* X *be a Banach space and* $F : [a, b] \to cwk(X)$ *be a measurable and integrably bounded multifunction. Then, if we set*

$$L = \int_a^b s(x', F(t))dt$$

for every $x' \in X'$ *we have*

$$\Phi(F, [a, b]) \subset \{z \in X : \ < x', z > \ \leq \ L\}. \tag{3}$$

Proof: Let $z \in \Phi(F, [a, b])$ and suppose that (3) is not true. Then $< x', z > -L = \alpha > 0$.
By definition of $(*)$-integral, there exists a gauge Δ_* such that, for every generalized McShane partition $(E_i, t_i)_i$ subordinate to $\Delta_*(\alpha/6)$, we have:

$$\limsup_n d\left(z, \sum_{i=1}^n F(t_i)\mu(t_i)\right) := r \leq \alpha/6. \tag{4}$$

Let now $\varepsilon > 0$ satisfy $r + \varepsilon < \alpha/3$. Then, in correspondence to ε, there exists an integer $\overline{n}$ such that, for every $n \geq \overline{n}$,

$$d\left(z, \sum_{i=1}^n F(t_i)\mu(E_i)\right) < \alpha/3.$$

Since F has weakly compact values, then, for every $n \geq \overline{n}$, there exists $x_n \in \sum_{i=1}^n F(t_i)\mu(E_i)$ such that $\|z - x_n\| = d(z, \sum_{i=1}^n F(t_i)\mu(E_i))$ and hence

$$\begin{aligned} < x', z > \ &\leq \ | < x', x_n > | + | < x', z - x_n > | \leq \qquad (5) \\ &\leq \ s(x', \sum_{i=1}^n F(t_i)\mu(E_i)) + \alpha/3 \leq \\ &\leq \ \sum_{i=1}^n s(x', F(t_i)\mu(E_i)) + \alpha/3. \end{aligned}$$

Moreover we know that $s(x', F)$ is Lebesgue integrable, since it is measurable and dominated by $h(F(t), \{0\})$ and so, by [7, Lemma 1J] already quoted, there exists a gauge Δ_0 such that for every generalized $\mathcal{P}_{\Delta_0}$ McShane partition $\Pi' = (E'_i, t'_i)_i$,

$$\sum_{i=1}^{n} s(x', F(t_i)\mu(E_i)) \leq L + \alpha/3. \tag{6}$$

Therefore, if we consider $\Delta = \Delta_* \cap \Delta_0$ and we take any generalized $\mathcal{P}_\Delta$ McShane partition, inequalities (4), (5) and (6) give us the following contradiction: $< x', z > \leq L + 2\alpha/3 =< x', z > -\alpha/3$. Hence (3) holds.□

Now we are in position to state our comparison result.

Theorem 1 *Suppose that X is a separable Banach space and that there exists a countable family $(x'_n)_n$ in X' which separates points of X. Then*

$$(A) \int_a^b F(t)dt = \Phi(F, [a, b])$$

holds, for any measurable and integrably bounded multifunction $F : [a, b] \to cwk(X)$.

Remark 2 Observe that this theorem extends [14, Theorem 3] in several directions: first of all, we obtain an analogous result in infinite dimensional spaces, rather then a Euclidean space. Moreover here multifunctions with unbounded domains are allowed, and their values are only requested to be convex and weakly-compact. The hypothesis of convexity of the values could not be dropped in our case; indeed in the infinite dimensional case there are examples of non convex Aumann integrals.

Proof of Theorem 1: The inclusion

$$(A) - \int_a^b F dt \subset \Phi(F, [a, b])$$

is contained in Proposition 2 which holds without any assumption on F and X. The other inclusion is proved similarly as in [14, Theorem

3]. Observe that from [4, Lemma III.32] it is possible to construct a countable family P which is dense in E' for $\tau(E', E)$. So, let $P = (x'_n)_n$. By [4, Lemma III.14] quoted above, we know that, for every $t \in [a, b]$,

$$F(t) = \bigcap_n \{z \in X : \ < x'_n, z > \ \leq \ s(x'_n, F(t))\} \,.$$

Set $L_n = \displaystyle\int_a^b s(x'_n, F(t))dt$. Applying Proposition 5 we have that

$$\Phi(F, [a, b]) \subset \bigcap_n \{z \in X : \ < x'_n, z > \ \leq \ L_n\} \,.$$

Observe also that, since F is $cwk(X)$-valued, then its (A)-integral belongs to the same hyperspace by Proposition 3 ([12, Proposition 5.20]) and then, using again [4, Lemma III.14], we have

$$(A) - \int_a^b F(t)dt = \bigcap_n \left\{ x \in X :< x'_n, x >\leq \ s(x'_n, (A) - \int_a^b F(t)dt) \right\} .$$

Thus we have proved that

$$\Phi(F, [a, b]) \subset (A) \int_a^b F(t)dt$$

and this concludes the proof of the theorem. □

Theorem 1 can be applied in the comparison of Aumann integral and other known integrals; for the relationship with the Debreu integral see for example [3, 19, 15, 16, 20]. For weakly compact valued multifunctions the result was obtained for totally measurable multifunctions. In general, measurable multifunctions are not totally measurable. Here we give an example of a measurable multifunction not totally measurable for which the Aumann and the ($\star$)-integrals coincide.

Example 1 Let $X = l^2(\mathbb{N}^*)$; for every $A \subset \mathbb{N}^*$ we consider

$$U_A = \{x \in X : \|x\| \leq 1, \text{ and } x_n = 0 \text{ if } \ n \notin A\} = \{1_A x : \|x\| \leq 1\},$$

where $(1_A x)_n = 1_A(n)x_n$. If $A \neq B$ then $h(U_A, U_B) \geq 1$ and so the set $\{U_A, A \subset \mathbb{N}^*\}$ is not separable.
Let $\Omega = [0,1[$ and for every $\omega \in \Omega$ let $0, \omega_1 \cdots \omega_n \cdots$ be its dyadic representation, namely $\omega_1 = 1$ iff $\omega \in [1/2, 1[$, $\omega_2 = 1$ iff $\omega \in [1/4, 1/2[\cup [3/4, 1[$, etc. We set $B_1 = [1/2, 1[, B_2 = [1/4, 1/2[\ \cup \ [3/4, 1[$, etc.
Let $F(\omega) = U_{A(\omega)}$ where $A(\omega) = \{n \in \mathbb{N}^* : \omega_n = 1\}$. F is integrably bounded, takes weakly compact and convex values and its support function $s(y, F(\omega))$ is measurable since it is the limit of simple functions; indeed:

$$s(y, F(\omega)) = \{\sum_{n \in A(\omega)} y_n^2\}^{1/2} = \lim_{n \to \infty} \sum_{p \leq n} s(y, F(\omega)) 1_{B_p(\omega)}$$

and

$$\sum_{p \leq n} s(y, F(\omega)) 1_{B_p(\omega)} = \{\sum_{p \leq n} y_p^2 : \omega_p = 1\}^{1/2}.$$

From [12, Proposition II.2.39] F is measurable, but for every μ-null set N, the set $\Omega \setminus N$ is not countable and so $F(\Omega \setminus N)$ is not separable in the h-metric topology. Then immediately it follows that F cannot be a member of the closure of simple multifunctions with weakly compact and convex values in the L^1-metric associated with h and so F is not a Bochner integrable multifunction. Moreover, by Theorem 1,

$$\Phi(F, [0,1[) = (A) - \int_0^1 F(\omega) d\mu(\omega).$$

4. The McShane multivalued integral

If we consider directly the hyperspace $(cwk(X), h)$ we can define the McShane multivalued integral in the following way:

Definition 3 We say that $F : [a,b] \to cwk(X)$ is *McShane integrable* if there exists $J \in cwk(X)$ such that for every $\varepsilon > 0$ there exists a gauge Δ such that

$$\limsup_n h(J, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \varepsilon$$

for every generalized $\mathcal{P}_\Delta$ McShane partition $\Pi = (E_i, t_i)_i$. In this case, we set

$$J := \int_a^b F(t)dt.$$

Thanks to the Rådstrom embedding theorem [18], this definition is well-posed, and we will show the following:

Theorem 2 *If $F : [a,b] \to cwk(X)$ is McShane integrable, then the $(\star)$-integral and the McShane integral coincide, namely $J = \Phi(F, [a,b])$.*

Proof: The inclusion $J \subset \Phi(F, [a,b])$ is obvious; indeed if $z \in J$, then for every $\varepsilon > 0$ there exists a gauge Δ such that for each generalized $\mathcal{P}_\Delta$ McShane partition $(E_i, t_i)_{i\in\mathbb{N}}$ we get:

$$\limsup_n d(z, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \limsup_n h(J, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \varepsilon.$$

Conversely, let now $z \in \Phi(F, [a,b])$. Then, for every $\varepsilon > 0$, there exists a gauge Δ such that, for every generalized $\mathcal{P}_\Delta$ McShane partition $(E_i, t_i)_{i\in\mathbb{N}}$, we have

$$\limsup_n d(z, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \varepsilon/2.$$

On the other hand, by the definition of the McShane integral, there exists a gauge Δ_1 such that for every generalized $\mathcal{P}_{\Delta_1}$ McShane partition $(E'_i, t'_i)_i$, we get

$$\limsup_n h(J, \sum_{i=1}^n F(t'_i)\mu(E'_i)) \leq \varepsilon/2.$$

So, if we take $\widetilde{\Delta} = \Delta \cap \Delta_1$, then, for every generalized $\mathcal{P}_{\widetilde{\Delta}}$ McShane partition $\Pi = (E_i, t_i)_i$ and for every $x \in \sum_{i=1}^n F(t_i)\mu(E_i)$ we have

$$\begin{aligned} d(z, J) &= \inf_{y \in J} \|z - y\| \leq \inf_{y \in J} (\|z - x\| + \|x - y\|) = \\ &= \|z - x\| + d(x, J) \leq \|z - x\| + h(\sum_{i=1}^n F(t_i)\mu(E_i), J). \end{aligned}$$

So we have

$$d(z, J) \leq \limsup_n \left(d(z, \sum_{i=1}^n F(t_i)\mu(E_i)) + h(\sum_{i=1}^n F(t_i)\mu(E_i), J) \right) \leq \varepsilon$$

for every generalized $\mathcal{P}_{\widetilde{\Delta}}$ McShane partition $(E_i, t_i)_i$. Since ε is arbitrary and $\Phi(F, [a, b])$ is closed, the last inclusion follows. □

Observe that if X is separable and $F : \Omega \to ck(X)$ is a measurable multifunction with unbounded range, then Debreu integrability implies McShane's one. In this case we can embed $(ck(X), h)$ in a suitable separable Banach space Y and, if we consider F as a Y-valued measurable function, McShane integrability coincides with the Pettis' one. If the range of F is bounded and $\mu(\Omega) < \infty$, the two concept of integral coincide; see for example [6, Section 2K].
If we consider a multifunction with weakly compact and convex values we need total measurability of F since $(cwk(X), h)$ is not separable in general. In this case we have:

Corollary 1 *If μ is finite, X is a separable Banach space and $F : \Omega \to cwk(X)$ is a Debreu integrable multifunction, then F is McShane integrable and its integral coincides with the Aumann integral of F.*

Proof: Thanks to the result of Byrne [3] the Debreu integral and the Aumann integral coincide; thank to [7, Theorem 1K] the Debreu and the McShane integrals coincide too. □

An example of an integrably bounded and McShane integrable multifunction which is not Debreu integrable can be obtained using [6, Example 3F] and taking $F(t) = \{f(t)\}$, where f takes values in the non separable Banach space $L^{\infty}([0,1])$.

Theorem 2 implies that, when F is $cwk(X)$-valued and McShane integrable, $\Phi(F,[a,b])$ is convex and weakly compact and so in this case we obtain [14, Proposition 1] as a corollary of Theorem 2.
Theorem 2 is also important from another point of view: indeed, thanks to the Rådstrom embedding theorem, all the fundamental results concerning the McShane integral, which are given in [7], are still valid for $cwk(X)$-valued multifunctions. So it is enough to consider the space $cwk(X)$ as a Banach space, which plays the role of the Banach space X in the previous section. So $\Phi(F,[a,b])$ satisfies the main fundamental properties of the functionals defined by means of integrals, like for example additivity and absolute continuity.

Remark 3 Though in this paper Ω is always assumed to be an interval in the real line (possibly unbounded), we observe that all the results here obtained hold as well whenever Ω is any non empty σ-finite quasi Radon outer regular measure space.

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