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A Radon-Nikodym theorem for multimeasures *

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Introduction

One of the most fascinating problems arising when dealing with finitely additive measures is that of the existence of a Radon-Nikodym derivative: as it is well known, the absolute continuity assumption is not sufficient in this case.
Several papers concerning this question appeared since the late sixties ([5, 6, 8, 11, 12, 13, 14]). When the dominated set function takes values in a vector space X, the investigation usually does not go beyond the case of Banach spaces (actually the only exact Radon-Nikodym Theorem for locally convex topological vector space valued finitely additive measures seems to be that in [6]).
The starting point of this note is the article of Castaing, Touzani and Valadier [8]: in it the authors obtain an elegant characterization of those multivalued finitely additive measures admitting approximated densities with respect to a scalar finitely additive measure. This approximated densities turn out to be in fact "simple" multifunctions.
Since every classical abstract integration theory makes use of simple functions as approximating tools ([10, 3]), it seemed rather natural to develop an integration theory for multifunctions F with closed convex bounded values in a locally convex topological vector space X, with respect to a finitely additive measure μ,

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analogously to that done in [10]; this is done in the second section of this paper. In the third section we obtain an exact Radon-Nikodym Theorem for such integration, under conditions of the classical Maynard-type. Our Theorem is new even for single valued finitely additive measures.

Handling the seminorms of X is quite difficult: indeed we had to request uniformity with respect to the seminorms for most of the conditions. In a further paper we shall investigate a milder type of integration in a locally convex topological vector space introduced by Blondia [1]. Nevertheless, when X is a Banach space, the assumptions are quite semplified, and the Radon-Nikodym Theorem becomes a multivalued version of that of Hagood. The authoresses are indebted to Professor Charles Castaing, who gave them several directions during this project; we also thank Professor J. Bán, who sent them connected papers and a large bibliography on the topic and Professor D. Candeloro who accurately read several versions of this paper.

1. Preliminaries and definitions

Let Ω be an abstract set and Σ a σ-algebra on Ω. Let X be a locally convex real linear space T_2; let Q be a family of continuous seminorms on X generating the topology of X, and let Q_0 be an absorbing subset of Q. We shall denote by $\mathcal{C}_c(X)$ the class of non empty, closed, convex and bounded subsets of X. For every continuous seminorm p let h_p denote the Hausdorff pseudo-distance generated by p on $\mathcal{C}_c(X)$. Then the family of seminorms $\{h_p(\cdot, \{0\}), p \in Q_0\}$ generates a locally convex topology on $\mathcal{C}_c(X)$ (see [8]). Let Y be a subspace of $\mathcal{C}_c(X)$ which is complete with respect to $\{h_p(\cdot, \{0\}), p \in Q_0\}$. Observe that if X is complete, the class $K(X)$ of non empty, compact, convex subsets of X fulfills this condition.

PROPOSITION 1.1 ([7]). *Y is T_2.*

For $A \in \mathcal{C}_c(X)$ let $\delta^*(\cdot|A) : X^* \to I\!R$ be the **support function**. For every $p \in Q_0$ let

$e_p : \mathcal{C}_c(X) \times \mathcal{C}_c(X) \rightarrow I\!R_0^+$ denotes the **excess** (namely $e_p(A,B) = \sup_{x\in A}\inf_{y\in B} p(x-y)$). We shall introduce some basic properties of the Hausdorff pseudo-distance that will be needed in the sequel.

PROPOSITION 1.2 ([7]). *Setting* $A + B = \overline{\{a+b, a \in A, b \in B\}}$ *for* $A, B \in \mathcal{C}_c(X)$

(1.2.a) $\delta^*(\cdot|A+B) = \delta^*(\cdot|A) + \delta^*(\cdot|B)$*;*

$\delta^*(\cdot|\lambda A) = \lambda\delta^*(\cdot|A)$ *for every* $\lambda \in I\!R_0^+$ *;*

(1.2.b) *if* $p \in Q$ *and* $U_p = \{x \in X |\ p(x) \le 1\}$, *denoted by* U_p^0 *its polar*

$$\begin{aligned} e_p(A,B) &= \sup\{\delta^*(x^*|A) - \delta^*(x^*|B) : x^* \in U_p^\circ\} \\ h_p(A,B) &= \sup\{|\delta^*(x^*|A) - \delta^*(x^*|B)| : x^* \in U_p^\circ\}; \end{aligned}$$

(1.2.c) *If* $A, B \in \mathcal{C}_c(X)$ *then* $h_p(A, A+B) = h_p(B, \{0\})$ *for every* $p \in Q$*;*

(1.2.d) *if* $A, B, C, D \in \mathcal{C}_c(X)$ *then for every* $p \in Q$

$$h_p(A+B, C+D) \le h_p(A,C) + h_p(B,D);$$

(1.2.e) *if* $A, B \in \mathcal{C}_c(X)$, A_n, B_n *are sequences in* $\mathcal{C}_c(X)$ *such that* $A = \sum_n A_n, B = \sum_n B_n$, *then for every* $p \in Q$, $h_p(A,B) \le \sum_n h_p(A_n, B_n)$*;*

(1.2.f) *for every* $t \in I\!R, p \in Q, A, B \in \mathcal{C}_c(X)$, $h_p(tA, tB) = |t| h_p(A,B)$.

DEFINITION 1.3. A set function $M : \Sigma \rightarrow Y$ is called a **finitely additive multimeasure** if for every $x^* \in X^*$ $\delta^*(x^*|M)$ is finitely additive, namely

$$\delta^*(x^*|M(A \cup B)) = \delta^*(x^*|M(A)) + \delta^*(x^*|M(B))$$

whenever $A, B \in \Sigma, A \cap B = \emptyset$. Then equivalently $M(A \cup B) = M(A) + M(B)$.

DEFINITION 1.4. Let $M : \Sigma \rightarrow Y$ be a finitely additive multimeasure. For every $p \in Q$ we define the p**-variation** of M as

$$|M|_p(E) = \sup_{(A_i)\in P(E)} \sum_{i\in I} h_p(M(A_i), \{0\})$$

for $E \in \Sigma$, where $P(E)$ denotes the family of finite Σ-measurable decompositions of E. We shall say that M is of **bounded variation (b.v.)** provided $|M|_p(X) < +\infty$ for every $p \in Q$.

DEFINITION 1.5. Let $M : \Sigma \to Y$, be a b.v. finitely additive multimeasure and $\mu : \Sigma \to I\!R$ a finitely additive measure. We shall say that M is **absolutely continuous** with respect to μ (and we shall write $M \ll \mu$) iff for every $\varepsilon > 0$ and every $p \in Q_0$ there exists $\delta(\varepsilon, p) > 0$ such that $|\mu|(E) < \delta$ yields $|M|_p(E) < \varepsilon$.

2. The integral of a multifunction

Let $\mu : \Sigma \to I\!R_0^+$ be a bounded finitely additive measure and $F : \Omega \to Y$ a multifunction. When F is simple, namely

$$F = \sum_{i=1}^{n} 1_{A_i} C_i, \qquad C_i \in Y \quad \forall\, i = 1, 2, \cdots, n$$

we define $\int_E F d\mu = \sum_{i=1}^{n} \mu(A_i \cap E) C_i$ as in [8]. Note that, since each C_i is convex, the integral does not depend upon the representation of F.

LEMMA 2.1. *If F, G are simple multifunctions, for every $p \in Q_0$*

$$h_p \left(\int_\cdot F d\mu, \int_\cdot G d\mu \right) \leq \int_\cdot h_p(F, G) d\mu. \tag{1}$$

LEMMA 2.2. *If F is a simple multifunction, and $M(E) = \int_E F d\mu$, for every $E \in \Sigma$ and for every $p \in Q_0$ $|M|_p(E) = \int_E h_p(F, \{0\}) d\mu$.*

Proof: the proof of the lemma is a straightforward transposition of its single valued analogous (see [10]).

DEFINITION 2.3. *F is* **"totally measurable"** *if there exists a sequence of simple multifunctions $(F_n)_n$ such that*

(i0) *$h_p(F_n, F)$ is measurable for every $p \in Q$ and for every $n \in I\!N$;*

(i1) *$h_p(F_n, F)$ μ-converges to zero for every $p \in Q$.*

We shall denote by $M[\Omega, X]$ the set of totally measurable multifunctions.

Observe that if F is totally measurable and G is simple then $h_p(F, G)$ is measurable for every $p \in Q$.

For $F \in M[\Omega, X]$ we shall introduce the definition of integral:

DEFINITION 2.4. Let $F : \Omega \to Y$ be totally measurable. F is μ-**integrable** iff there exists a sequence $(F_n)_n$ of simple multifunctions satisfying **(i0)**, **(i1)** and such that, for every $p \in Q$:

(i2) $\lim_{m,n\to\infty} \int_\Omega h_p(F_n, F_m) d\mu = 0$.

We shall say that $(F_n)_n$ is a **defining sequence** for F.

Then for every $E \in \Sigma$ the sequence $\left(\int_E F_n d\mu\right)_n$ is Cauchy in Y and therefore it converges in Y. We then set

$$\int_E F d\mu = \lim_{n\to\infty} \int_E F_n d\mu.$$

Observe that if X is complete and $Y = \{ \{c\}, c \in X\}$ the above definition coincides with the usual definition of μ-integrable, single valued functions ([10]).

Moreover if F is μ-integrable, $\int F d\mu$ is a multimeasure.

REMARK 2.5.

(2.5.a) If F is μ-integrable and $(F_n)_n$ is a defining sequence of simple multifunctions then, for every $p \in Q_0$, the functions $h_p(F_n, \{0\})$ are uniformly integrable;

(2.5.b) for every $n \in I\!N$ the functions $\{h_p(F_m, F_n)\}_m$, which are measurable and non negative, are uniformly μ-integrable;

(2.5.c) $\lim_{n\to\infty} \int_\Omega h_p(F_n, F) d\mu = 0$;

(2.5.d) The function $h_p(F, \{0\})$ is μ-integrable and $\int_E h_p(F, \{0\}) d\mu = \lim_{n\to\infty} \int_E h_p(F_n, \{0\}) d\mu$.

We observe that by **(1)** if F is a simple multifunction

$$h_p\left(\int_\cdot F d\mu, \{0\}\right) \leq \int_\cdot h_p(F, \{0\}) d\mu.$$

We want now extend such property to every μ-integrable multifunction.

THEOREM 2.6. *Let F be μ-integrable; then for every $p \in Q_0$*

$$h_p\left(\int_{\cdot} F d\mu, \{0\}\right) \leq \int_{\cdot} h_p(F, \{0\}) d\mu;$$

moreover, for every $p \in Q_0$, $|M|_p(\cdot) = \int_{\cdot} h_p(F, \{0\}) d\mu$.

Proof: Let $(F_n)_n$ be a defining sequence for F. Then

$$\lim_{n\to\infty} h_p\left(\int_E F_n d\mu, \{0\}\right) = h_p\left(\int_E F d\mu, \{0\}\right).$$

By **(2.5.c)** it is

$$\begin{aligned} h_p\left(\int_E F d\mu, \{0\}\right) &= \lim_{n\to\infty} h_p\left(\int_E F_n d\mu, \{0\}\right) \leq \lim_{n\to\infty} \int_E h_p(F_n, \{0\}) d\mu \leq \\ &\leq \int_E h_p(F, \{0\}) d\mu. \end{aligned}$$

We shall prove now that $|M|_p(\cdot) = \int_{\cdot} h_p(F, \{0\}) d\mu$. Let $M_n(E) = \int_E F_n d\mu$ and $\varepsilon > 0$ be fixed; there exists $\overline{n}$ such that for every $n \geq \overline{n}$

$$\int_E h_p(F_n, F) d\mu < \varepsilon.$$

We can choose $A_1, \cdots A_m, B_1, \cdots, B_q$ such that

$$|M|_p(E) - \sum_{i=1}^{n} h_p\left(\int_{A_i} F d\mu, \{0\}\right) < \varepsilon$$

$$|M_n|_p(E) - \sum_{i=1}^{q} h_p\left(\int_{B_i} F_n d\mu, \{0\}\right) < \varepsilon.$$

Then let $(E_i)_i$ be the decomposition of E generated by the sets A_i, B_j. It is $|\ |M|_p(E) - |M_n|_p(E)| < 3\varepsilon$ and so

$$|M|_p(E) = \int_E h_p(F, \{0\}) d\mu.$$

THEOREM 2.7. *If F is μ-integrable then $\int_{\cdot} F d\mu \ll \mu$.*

Proof: It is a consequence of **(2.5.c)** and Theorem 2.6.

We shall now prove that the integral is well defined.

THEOREM 2.8. *Let $(F_n)_n$, $(G_n)_n$ be two sequences of simple multifunctions defining F. Then it is*

$$\int_E F d\mu = \lim_{n\to\infty} \int_E F_n d\mu = \lim_{n\to\infty} \int_E G_n d\mu \quad \forall\ E \in \Sigma.$$

Proof: By inclusion one easily shows that $h_p(F_n, G_n)$ μ-converges to 0 for every $p \in Q$. Let $\varepsilon > 0$ be fixed and $A_n = \{x \in \Omega : h_p(F_n(x), G_n(x)) > \varepsilon\}$.
We shall prove that $\lim_{n\to\infty} h_p(\int_E F_n d\mu, \int_E G_n d\mu) = 0$ for all $E \in \Sigma$. For every $n > \overline{n}$ suitably large

$$\begin{aligned} h_p\left(\int_E F_n d\mu, \int_E G_n d\mu\right) &\leq \int_E h_p(F_n, G_n) d\mu = \int_{E\cap A_n} h_p(F_n, G_n) d\mu + \\ &+ \int_{E-A_n} h_p(F_n, G_n) d\mu \leq \int_{E\cap A_n} h_p(F_n, G_n) d\mu + \varepsilon\mu(\Omega). \end{aligned}$$

It only remains to prove that $\lim_{n\to\infty} \int_{E\cap A_n} h_p(F_n, G_n) d\mu = 0$. This is a standard proof, based upon the Vitali-Hahn-Saks Theorem [1] (see for example [3]).

A finite or countable family of pairwise disjoint sets $(E_i)_i$, $E_i \in \Sigma$ will be called a μ-**exhaustion** of Ω provided for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $|\mu|(\Omega - \cup_{i\leq n} E_i) < \varepsilon$. Let $\{E_i\}_i$ be a μ-exhaustion of Ω such that $\Omega = \bigcup_{i\in\mathbb{N}} E_i$. Let $(C_n)_n$ be a sequence in Y such that for every $p \in Q$ there exists $r_p > 0$, such that $h_p(C_n, \{0\}) \leq r_p$ for every n. We define

$$\textbf{(*)} \qquad F = \sum_{i=1}^{\infty} C_i 1_{E_i};$$

such definition is formal, in the sense that for every $x \in \Omega$ there exists a unique $i = i(x) \in \mathbb{N}$ such that $x \in E_{i(x)}$ and so $F(x) = C_{i(x)}$. Hence F is Y valued.

PROPOSITION 2.9. *F defined as in* **(*)** *is μ-integrable.*

Proof: We prove that $F_n = \sum_{i=1}^{n} C_i 1_{E_i}$ is defining for F. As noted above, for every $x \in \Omega$ $\exists!$ $i(x)$ such that $F(x) = F_n(x)$ for every $n \geq i(x)$ and, for every $\varepsilon > 0$, the set $A_\varepsilon = \{x \in \Omega : h_p(F_n(x), F(x)) > \varepsilon\} = \bigcup_{i=n+1}^{\infty} \{x \in E_i : h_p(C_i, \{0\}) > \varepsilon\} \in \Sigma$ and moreover $A_\varepsilon \subset \bigcup_{i=n+1}^{\infty} E_i$; since $\{E_i\}_{i\in\mathbb{N}}$ is a μ-exhaustion, by the monotonicity of μ, F_n μ-converges to F uniformly with respect to $p \in Q$. For $m > n$ we have

$$\int_\Omega h_p(F_n, F_m) d\mu \ = \ \textstyle\sum_{i=n+1}^{m} \mu(E_i) h_p(C_i, \{0\}) \leq r_p \displaystyle\sum_{i=n+1}^{m} \mu(E_i)$$

and since $\{E_i\}_i$ is a μ-exhaustion, this proves the integrability.

DEFINITION 2.10. A multifunction $F : \Omega \to Y$ is **Q_0-strongly μ-integrable** if there exists a sequence of simple multifunctions $(F_n)_n$ such that, uniformly with respect to $p \in Q_0$

(i1*) F_n μ-converges to F;

(i2*) $\lim_{m,n\to\infty} \int_\Omega h_p(F_n, F_m) d\mu = 0.$

REMARK 2.11. If $F = \sum_{i=1}^{\infty} C_i 1_{E_i}$, where $\{E_i\}_i$ is a μ-exhaustion of Ω and $(C_n)_n$ is such that there is $r > 0$ for which $h_p(C_n, \{0\}) \le r$ for all $p \in Q_0$ and for all $n \in I\!N$, then **(i1*)** and **(i2*)** hold.

PROPOSITION 2.12. *Let F, G be μ-integrable. Then for each $p \in Q_0$*

$$h_p\left(\int_E F d\mu, \int_E G d\mu\right) \le \int_E h_p(F,G) d\mu, \quad E \in \Sigma.$$

Proof: Let $(F_n)_n, (G_n)_n$ be sequences of simple multifunctions defining F and G respectively. Then for n suitable large

$$h_p\left(\int_E F d\mu, \int_E G d\mu\right) \le 2\varepsilon + \int_E h_p(F_n, G_n) d\mu \tag{2}$$

Since

$$|h_p(F,G) - h_p(F_n, G_n)| \le h_p(F, F_n) + h_p(G, G_n)$$

$h_p(F,G)$ is the μ-limit of simple functions, and hence it is measurable. Also $h_p(F_n, G_n)_n$ is Cauchy in $L^1(\Omega, \Sigma, \mu)$, since

$$\int_\Omega |h_p(F_m, G_m) - h_p(F_n, G_n)| d|\mu| \le \int_\Omega h_p(F_n, F_m) d|\mu| + \int_\Omega h_p(G_n, G_m) d|\mu|;$$

therefore $h_p(F,G)$ is μ-integrable and

$$\lim_{n\to\infty} \int_\Omega h_p(F_n, G_n) d\mu = \int_\Omega h_p(F,G) d\mu;$$

By taking the limit in **(2)** then the assertion follows.

Let now μ be a real valued finitely additive measure of bounded variation; then F is **μ-integrable** iff F is $|\mu|$-integrable and we shall put

$$\int F d\mu = \int F d\mu^+ - \int F d\mu^-.$$

PROPOSITION 2.13. *If F is μ-integrable then for every $p \in Q$*

$$h_p\left(\int_E F d\mu, \{0\}\right) \leq \int_E h_p(F, \{0\}) d|\mu|;$$
$$|M|_p(E) = \int_E h_p(F, \{0\}) d|\mu|.$$

Proof: The proof of the first assertion is straightforward. The second equality can be proven analogously to what already done in Theorem 2.6.

We shall give now the main convergence Theorem.

THEOREM 2.14 (Vitali Convergence Theorem). *Let $(F_n)_n$ be a sequence of Q_0-strongly μ-integrable multifunctions such that for every $p \in Q_0$:*

(v1) *F_n μ-converges to F;*

(v2) *$\int F_n d\mu \ll \mu$ uniformly with respect to n;*

then F is μ-integrable and $\int_. F d\mu = \lim_{n\to\infty} \int_. F_n d\mu$.

Proof: By inclusion one easily shows that

$$\lim_{m,n\to\infty} |\mu|(\{x \in \Omega : h_p(F_n(x), F_m(x)) > \varepsilon\}) = 0$$

for every $\varepsilon > 0$, $p \in Q_0$. Also, from **(v2)** one obtains that $(\int_E F_n d\mu)_n$ is Cauchy in Y for every $E \in \Sigma$. To show that F is μ-integrable, we have to determine a defining sequence of simple multifunctions $(G_n)_n$. By assumption, for each n there exists a sequence $(G_k^{(n)})_{k\in\mathbb{N}}$ of simple multifunctions such that, uniformly with respect to $p \in Q_0$, $G_k^{(n)}$ μ-converges to F_n, and $(h_p(G_k^{(n)}, F_n))_k$ converges to 0 in $L^1(\Omega, \Sigma, \mu)$. If $H^p_{n,k}$ denotes the set

$$H^p_{n,k} = \{x \in \Omega : h_p(G_k^{(n)}(x), F_n(x)) > \frac{1}{2^n}\},$$

we obtain that, uniformly with respect to $p \in Q_0$, for every $n \in \mathbb{N}$ there is $\overline{k}(n)$ such that for every $k \geq \overline{k}$ $|\mu|(H^p_{n,k}) < \frac{1}{2^n}$ and

$$\int_\Omega h_p(G_k^{(n)}, F_n) d|\mu| \leq \frac{1}{2^n}.$$

Let $B_n^p = H^p_{n,\overline{k}_n}$. Then $\lim_{n\to\infty}|\mu|(B_n^p) = 0$ uniformly with respect to $p \in Q_0$, and if we set $G_n = G^{(n)}_{\overline{k}_n}$, for $x \in (B_n^p)^c$ $h_p(G_n(x), F(x)) \leq \dfrac{1}{2^n}$. Then G_n μ-converges to F; in fact, by the triangular inequality

$$|h_p(G_m, G_n) - h_p(G_m, F)| \leq h_p(G^{(n)}_{\overline{k}_n}, F_n) + h_p(F_n, F)$$

which shows that $h_p(G_n, F)$ is totally measurable; furthermore it μ-converges to 0.

We shall now prove that, for $p \in Q_0$,

$$\int_\Omega h_p(G_n, G_k) d|\mu| \to 0 \text{ for } n, k \to \infty.$$

Since, for n, k suitable large,

$$\begin{aligned} \int_\Omega h_p(G_n, F_n) d|\mu| &\leq \frac{1}{2^n} \\ \int_\Omega h_p(G_k, F_k) d|\mu| &\leq \frac{1}{2^k} \end{aligned}$$

and, by assumption **(v2)**,

$$\int_\Omega h_p(F_n, F_k) d|\mu| \leq \varepsilon$$

one finds, with a standard decomposition of Ω,

$$\int_\Omega h_p(G_n, G_k) d|\mu| \leq \varepsilon + \frac{1}{2^n} + \frac{1}{2^k}.$$

Thus F is μ-integrable. Moreover $h_p(\int_E F_n d\mu, \int_E G_n d\mu) \to 0$ for every $p \in Q, E \in \Sigma$. In fact

$$h_p\left(\int_E G_n d\mu, \int_E F_n d\mu\right) \leq \int_\Omega h_p(G_n, F_n) d|\mu| \leq \frac{1}{2^n}$$

therefore

$$\int_E F d\mu = \lim_{n\to\infty} \int_E G_n d\mu = \lim_{n\to\infty} \int_E F_n d\mu.$$

The proof is now complete.

As a consequence we find

THEOREM 2.15 (Lebesgue Dominated Convergence Theorem). *Let $(F_n)_n$ be a sequence of Q_0-strongly μ-integrable multifunctions μ-converging to F, and such* *that there exists* $g \in L^1(\Omega, \Sigma, \mu)$ *with* $h_p(F_n(x), \{0\}) \leq g(x)$ *for* $x \in \Omega, n \in I\!N, p \in Q_0$. *Then* F *is* μ*-integrable and for* $E \in \Sigma$

$$\int_E F d\mu = \lim_{n\to\infty} \int_E F_n d\mu.$$

LEMMA 2.16. *Let* $(G_n)_n$ *be a sequence of* Q_0*-strongly* μ*-integrable multifunctions that converges to* G*, uniformly with respect to* $x \in \Omega$ *and* $p \in Q_0$. *Then* G *is* Q_0*-strongly* μ*-integrable and*

$$\int_E G d\mu = \lim_{n\to\infty} \int_E G_n d\mu.$$

Proof: The proof is analogous to that of Theorem 2.14. In fact, if $(\Gamma_k^{(n)})_k$ is a sequence of simple multifunctions defining G_n (uniformly with respect to $p \in Q_0$), for every $\alpha > 0, n \in I\!N$ there exists $k(n,\alpha)$ such that, for $k > k(n,\alpha)$

$$\{x \in \Omega : h_p(\Gamma_k^{(n)}(x), G_n(x)) > \alpha\} = \emptyset$$

and, for n suitable large

$$\{x \in \Omega : h_p(G(x), G_n(x)) > \alpha\} = \emptyset$$

uniformly with respect to $p \in Q_0$. Then, for every n, taking $\Gamma_n = \Gamma^{(n)}_{k(n,\frac{1}{2^n})}$ one obtains a sequence that defines G uniformly with respect to $p \in Q_0$.

3. A Radon-Nikodym Theorem

DEFINITION 3.1. A finitely additive multimeasure $M : \Sigma \to Y$ is **bounded** if it has bounded range $R(M)$, namely for every $p \in Q_0$ there exists $L_p \in I\!R^+$ such that $\sup_{A\in\Sigma} h_p(M(A), \{0\}) \leq L_p$. In this case we shall put $Q_M = \{q : q = \frac{p}{L_p}, p \in Q\}$. We shall say that M is Q_M**-uniformly bounded** when L does not depend on p.

Let $\mu : \Sigma \to I\!R$ be a bounded finitely additive measure and let $|\mu|$ be its variation. With Σ^+ and Σ^2 we shall denote the subsets of Σ defined by:

$$\Sigma^+ = \{E \in \Sigma : |\mu|(E) > 0\}; \qquad \Sigma^2 = \{E \in \Sigma : |\mu|(E) < 2|\mu(E)|\}.$$

With the symbol $E\Sigma$ we shall denote $E \cap \Sigma$, and analogous meaning will have the symbols $E\Sigma^+,\ E\Sigma^2$. We shall now introduce the following "ranges" of M with respect to μ:

$$\begin{aligned} A(E\Sigma^2) &= \left\{ \frac{M(F)}{\mu(F)}, F \in E\Sigma^2, \mu(F) \neq 0 \right\}, \\ A_p(E,\varepsilon) &= \{C \in Y : h_p(M(F), C\mu(F)) \leq \varepsilon|\mu|(F)\ \forall\ F \in E\Sigma\} \\ A(E,\varepsilon) &= \bigcap_{p \in Q_M} A_p(E,\varepsilon) \end{aligned}$$

that will be called **average range ,** (ε, p)**-approximated range** and ε-**approximated range** respectively.

DEFINITION 3.2. We shall say that a property **P** is **exhaustive** on a set $E \in \Sigma$ if there exists a μ-exhaustion $(E_i)_i,\ E_i \in E\Sigma$ such that for every i E_i fulfills the **P** property. In this case $(E_i)_i$ will be also referred to as a **P-exhaustion**.
A property **P** is **null difference** whenever for any pair of sets $A, B \in \Sigma^+$ such that $|\mu|(A\Delta B) = 0$ either both satisfy the **P** property or neither does.

LEMMA 3.3. *If $M \ll \mu$ then the property $A(E,\varepsilon) \neq \emptyset$ is "null difference"*

Proof: We shall show that if $E, F \in \Sigma^+$ and $|\mu|(E\Delta F) = 0$ then $A(E,\varepsilon) = A(F,\varepsilon)$ for all $\varepsilon > 0$. Note first that, since $M \ll \mu$, $h_p(M(E\Delta F), \{0\}) = 0$ for $p \in Q_M$.
If $C \in A(E,\varepsilon)$, namely $C \in A_p(E,\varepsilon)$ for each $p \in Q_M$, and $H \subset F$ then

$$H = (H \cap E) \cup (H \cap E^c) \subset (H \cap E) \cup (F \cap E^c),$$

and since $|\mu|(F - E) = 0$, it follows $|\mu|(H) = |\mu|(H \cap E)$. To prove that $C \in A_p(F,\varepsilon)$ note that by **(1.2.c)** and by the triangular property

$$h_p(M(H), C\mu(H)) \ = \ h_p(M(H \cap E), C\mu(H \cap E)) \leq \varepsilon|\mu|(H \cap E) \leq \varepsilon|\mu|(H).$$

whence $C \in \bigcap_{p \in Q_M} A_p(F,\varepsilon)$. Analogously one can show the converse inclusion.

The following result will be used in the sequel.

LEMMA 3.4 ([13]). *If $\mu : \Sigma \to I\!R$ is a bounded finitely additive measure and $E \in \Sigma^+$ then either $E \in \Sigma^2$ or there exists $F \subset E$ such that $F \in \Sigma^2$.*

DEFINITION 3.5. Let $\mathcal{C} \subset Y$; we shall define, for $p \in Q$, the p-**diameter** of $\mathcal{C}$ as the number $\delta_p(\mathcal{C}) = \sup_{C,D \in \mathcal{C}} h_p(C, D) \leq 2 \sup_{C \in \mathcal{C}} h_p(C, \{0\})$.
A set $\mathcal{C}$ is Q_0-**uniformly bounded** if $\sup_{p \in Q_0} \sup_{C \in \mathcal{C}} h_p(C, \{0\}) < +\infty$.

LEMMA 3.6. *Let $M : \Sigma \to Y$ and $\mu : \Sigma \to I\!R$ be bounded finitely additive measures such that:*

(3.6.1) *$M \ll \mu$;*

(3.6.2) *$A(\Omega\Sigma^2)$ is Q_M-uniformly bounded;*

(3.6.3) *for every $\varepsilon > 0$ the property $A(E, \varepsilon) \neq \emptyset$ is exhaustive on any $E \in \Sigma^+$.*

Then there exists a multifunction $G : \Omega \to Y$ Q_M-strongly μ-integrable, Q_M-uniformly bounded (i.e. with Q_M-uniformly bounded range) such that

$$\int_E G d\mu = M(E) \qquad \text{for every } E \in \Sigma.$$

Proof: Analogously to what is done in [14], [11], [13] we can obtain a sequence of μ-exhaustions of Ω $(E^n_\alpha)_n, \alpha \in I\!N^n$ such that:

(3.6.4) $A(E^n_\alpha, 2^{-n}) \neq \emptyset$ for each $n \in I\!N$ and $\alpha \in I\!N^n$;

(3.6.5) $E^n_\alpha = \cup_i E^{n+1}_{\alpha,i}$ where $(E^{n+1}_{\alpha,i})_i$ is a μ-exhaustion of E^n_α for each $n \in I\!N$ and $\alpha \in I\!N^n$;

(3.6.6) for each fixed $n \in I\!N$ and $\alpha \in I\!N^n \Omega = \cup_\alpha E^n_\alpha$.

Defining

$$G_n = \sum_\alpha C^n_\alpha 1_{E^n_\alpha} \qquad C^n_\alpha \in A(E^n_\alpha, 2^{-n}),$$

according to Remark 2.11, G_n is Q_M-strongly μ-integrable. This is easily shown by means of Lemma 3.4, since $h_p(C^n_\alpha, \{0\}) \leq 1 + 2L$ for every $\alpha \in I\!N^n, n \in I\!N$, where

$$L = \sup_{p \in Q_M} \sup_{K \in \Omega\Sigma^2} h_p(\frac{M(K)}{\mu(K)}, \{0\}).$$

We shall then show that $(G_n)_n$ is Cauchy in Y, uniformly with respect to $x \in \Omega$ and $p \in Q_M$. Observe first that, for $m, n \in I\!N$ fixed, $n > m$, as done in [11] one can write

$$G_m = \sum_{(\alpha,\beta)} C^m_{(\alpha,\beta)} \cdot 1_{E^m_{(\alpha,\beta)}} \qquad G_n = \sum_{(\alpha,\beta)} C^n_{(\alpha,\beta)} \cdot 1_{E^n_{(\alpha,\beta)}}$$

for $\alpha \in I\!N^m, \beta \in I\!N^{n-m}, C^m_{(\alpha,\beta)}, C^n_{(\alpha,\beta)} \in A(E^m_\alpha, 2^{-m})$. From **(1.2.e)**

$$\begin{aligned} h_p(G_m, G_n) &= h_p\left(\sum_{(\alpha,\beta)} C^m_{(\alpha,\beta)} \cdot 1_{E^m_{(\alpha,\beta)}}, \sum_{(\alpha,\beta)} C^n_{(\alpha,\beta)} \cdot 1_{E^n_{(\alpha,\beta)}}\right) \leq \\ &\leq \sum_{(\alpha,\beta)} h_p(C^m_{(\alpha,\beta)}, C^n_{(\alpha,\beta)}) \cdot 1_{E^m_{(\alpha,\beta)}} \leq 2^{2-m}, \end{aligned} \tag{3}$$

Hence the sequence $(G_n)_n$ is Cauchy, uniformly with respect to $x \in \Omega$ and $p \in Q_M$. Let $G(x) = \lim_{n\to\infty} G_n(x)$. From **(3)** and by the way G is defined

$$h_p(G(x), \{0\}) \leq 1 + 2L \qquad \forall\, p \in Q_M,\ \forall x \in \Omega.$$

From Lemma 2.16 G is Q_M-strongly μ-integrable, and for every $E \in \Sigma$

$$\lim_{n\to\infty} h_p\left(\int_E G d\mu, \int_E G_n d\mu\right) = 0.$$

Finally we prove that, for all $E \in \Sigma$ and $p \in Q_M$, $h_p(M(E), \int_E G d\mu) = 0$.
Let $p \in Q_M$ and $\varepsilon > 0$ be fixed; for every $E \in \Sigma$ and $n \in I\!N$ fixed, the family $(E \cap E^n_\alpha)_{\alpha \in I\!N^n}$ is a a μ-exhaustion of E. Let $q \in I\!N$ be such that for every n-thuple $\alpha < (q, \cdots, q)$

$$|\mu|(E - \bigcup_{\alpha<(q,\cdots,q)} E \cap E^n_\alpha) < \delta(\frac{\varepsilon}{4}, p)$$

where δ is determined by the absolute continuity assumption.
Let $\overline{n}_p \in I\!N$ be such that for every $n > \overline{n}_p$

$$h_p\left(\int_E G_n d\mu, \int_E G d\mu\right) < \frac{\varepsilon}{4}$$

and

$$h_p\left(\sum_{\alpha<(q,\cdots,q)} C^n_\alpha \mu(E \cap E^n_\alpha), \int_E G_n d\mu\right) < \frac{\varepsilon}{4}$$

Analogously to [13] Theorem 3.1, one finds

$$h_p(\sum_{\alpha<(q,\cdots,q)} M(E \cap E^n_\alpha), \sum_{\alpha<(q,\cdots,q)} C^n_\alpha \mu(E \cap E^n_\alpha)) \leq \sum_{\alpha<(q,\cdots,q)} 2^{-n} |\mu|(E \cap E^n_\alpha)$$

and hence, for every $n > \overline{n}_p$

$$h_p(M(E), \int_E G d\mu) \leq \frac{3\varepsilon}{4} + \sum_{\alpha<(q,\cdots,q)} 2^{-n} |\mu|(E \cap E^n_\alpha)) \leq \frac{3\varepsilon}{4} + \frac{1}{2^n} |\mu|(E).$$

DEFINITION 3.7. *A* **P** *property is said to be* **hereditary** *if for each* $A, B \in \Sigma^+$ *with* $B \subseteq A$ *if* A *satisfies* **P** *then* B *satisfies* **P** *also.*

Observe that the property of having non empty ε-approximated range is hereditary. Analogously the property of having small average range is hereditary, see [14] for the definition.

We mention a result due to [11] that we shall use later.

PROPOSITION 3.8 (Exhaustion principle). *Let* $\mu : \Sigma \to I\!R$ *be a bounded finitely additive measure. The following conditions are equivalents:*

(E.1) *the hereditary property* **P** *is a* μ*-exhaustion on every* $E \in \Sigma^+$*;*

(E.2) *for every* $\delta > 0$*, there exist* $C \in \Sigma^+$ *and* $\alpha \in]0, 1[$ *such that:*

- **(E.2.a)** $|\mu|(\Omega - C) < \delta$*;*
- **(E.2.b)** *for every* $E \in C\Sigma^+$ *there exists* $F \in E\Sigma^+$ *such that* $|\mu|(F) > \alpha|\mu|(E)$ *and* F *satisfies* **P***.*

In this case we will say that **P** *is* **locally exhaustive***.*

The converse of Lemma 3.6 is not true: in fact from the existence of a density one cannot obtain **(3.6.3)**; it is only possible to obtain that the property of having non empty (ε, p)-approximated range is exhaustive for $p \in Q_M$. But if we strengthen the hypothesis we can obtain the following Lemma:

LEMMA 3.9. *Let* $M : \Sigma \to Y$ *be a bounded finitely additive multimeasure and* $\mu : \Sigma \to I\!R$ *a finitely additive measure. Let* $G : \Omega \to Y$ *be a* Q_M*-uniformly bounded* μ*-integrable multifunction which is the limit, uniformly with respect to* $p \in Q_M$ *and* $x \in \Omega$*, of* Q_M*-strongly* μ*-integrable multifunctions such that*

$$\int_E G d\mu = M(E) \qquad\qquad \textit{for every } E \in \Sigma;$$

then **(3.6.1)***,* **(3.6.2)** *and* **(3.6.3)** *are satisfied.*

Proof: Property **(3.6.1)** is obvious. We now prove that $A(\Omega\Sigma^2)$ is Q_M-uniformly bounded. Let $F \in \Omega\Sigma^2$ be such that $|\mu(F)| \neq 0$; we have that:

$$h_p(\frac{M(F)}{\mu(F)}, \{0\}) \ = \ h_p \left(\frac{\int_F G d\mu}{\mu(F)}, \{0\} \right) \leq \frac{1}{|\mu(F)|} \int_F h_p(G, \{0\}) d|\mu| \leq \frac{r|\mu|(F)}{|\mu(F)|} \leq 2r$$

for every $p \in Q_M$. This proves **(3.6.2)**. Finally we verify **(3.6.3)**. By the assumptions and by Lemma 2.16 there exists a sequence $(G_n)_n$ of μ-integrable simple multifunctions such that G_n converges to G uniformly with respect to $p \in Q_M$ and $x \in \Omega$. Let $\varepsilon > 0$ be fixed; then there exists $\overline{n}$ such that for every $n \geq \overline{n}$ $h_p(G(x), G_n(x)) < \varepsilon$ for every $x \in \Omega$ and $p \in Q_M$. Let $G_{\overline{n}} = \sum_{i=1}^k C_i 1_{E_i}$ and $\alpha = \dfrac{1}{2k}$. Let $E \in \Sigma^+$; then $|\mu|(E) = \sum_{i=1}^k |\mu|(E \cap E_i)$ and then there exists $j \in \{1, 2, \cdots, k\}$ such that $|\mu|(E \cap E_j) > \dfrac{|\mu|(E)}{2k}$. Setting $F = E \cap E_j$ it is $|\mu|(F) > \alpha|\mu|(E)$ and $A(F, \varepsilon) \neq \emptyset$. In fact for every $B \in F\Sigma^+$ and $p \in Q_M$

$$h_p(M(B), C_j \mu(B)) \ = \ h_p \left(\int_B G d\mu, \int_B G_{\overline{n}} d\mu \right) \leq \int_B h_p(G, G_{\overline{n}}) d|\mu| \leq \varepsilon|\mu|(B).$$

PROPOSITION 3.10. *Let M and μ be as above; the following two statements are equivalent:*

(3.10.1) *for every $E \in \Sigma^+$ and $\varepsilon > 0$ $A(E, \varepsilon) \neq \emptyset$ yields that $\delta(A(E\Sigma^2)) \leq 2\varepsilon$;*

(3.10.2) *for every $E \in \Sigma^+$ and $\varepsilon > 0$ $\delta(A(E\Sigma^2)) \leq \varepsilon$ yields that $A(E, \varepsilon) \neq \emptyset$.*

Proof: let $\varepsilon > 0$ and $E \in \Sigma^+$ be fixed in such a way that $A(E, \varepsilon) \neq \emptyset$. Let $C \in A(E, \varepsilon)$; then for every $p \in Q_M$

$$\sup_{F \in E\Sigma^2, \mu(F) \neq 0} h_p \left(\frac{M(F)}{\mu(F)}, C \right) \leq 2\varepsilon$$

and hence $\delta[A(E\Sigma^2)] < 2\varepsilon$.
Conversely let $G \in E\Sigma^2$, $C = \dfrac{M(G)}{\mu(G)}$ and $F \in E\Sigma^2$; then for every $p \in Q_M$

$$h_p(M(F), C\mu(F)) = |\mu(F)| \cdot h_p \left(\frac{M(F)}{\mu(F)}, \frac{M(G)}{\mu(G)} \right) \leq \varepsilon|\mu|(F)$$

and therefore $C \in A_p(E, \varepsilon)$ for all $p \in Q_M$. Hence, by the Exhaustion Principle we have the equivalence between the local exhaustivity of these two properties.

THEOREM 3.11 (Radon-Nikodym). *Let $M : \Sigma \to Y$ be a b.v. finitely additive multimeasure and $\mu : \Sigma \to I\!R$ a bounded finitely additive measure. Then the following are equivalent:*

(RN.1) *there exists a μ-integrable multifunction $G : \Omega \to Y$ satisfying:*

(RN.1a) *there is a μ-exhaustion $(E_i)_i$ of Ω such that for every i $G : E_i \to Y$ is the limit, uniformly with respect to $p \in Q_M$ and $x \in E_i$, of Q_M-strongly μ-integrable multifunctions;*

(RN.1b) *for every i there is $r_i > 0$ such that $\sup_{p\in Q_M} h_p(G(x), \{0\}) \leq r_i$ for all $x \in E_i$;*

(RN.1c) $\int_E G d\mu = M(E)$ *for every $E \in \Sigma$;*

(RN.2)

(RN.2a) $M \ll \mu$*;*

(RN.2b) *for every $\varepsilon > 0$ and $\delta > 0$ there are $C \in \Sigma^+$ and $\alpha \in]0, 1[$ such that:*

(RN.2b.1) $|\mu|(\Omega - C) < \delta$*;*

(RN.2b.2) *$A(C\Sigma^2)$ is Q_M-uniformly bounded;*

(RN.2b.3) *for every $E \in C\Sigma^+$ there is $F \in E\Sigma^+$ with $|\mu|(F) > \alpha|\mu|(E)$ and $A(F, \varepsilon) \neq \emptyset$.*

Proof: Let us first prove the implication **(RN.1)** $\Rightarrow$ **(RN.2)**. Firstly $M \ll \mu$. Since $(E_i)_i$ is a μ-exhaustion of Ω, for every $\delta > 0$ there shall be $n_1(\frac{\delta}{2}) \in I\!N$ with $|\mu|(\Omega - \cup_{i=1}^{n_1} E_i) < \dfrac{\delta}{2}$. Applying Lemma 3.9 to each $E_i, i = 1, \cdots, n_1$, for $\varepsilon > 0$ and $\dfrac{\delta}{2n_1}$ there shall exist $K_i \in E_i\Sigma^+$ and $\alpha_i \in]0, 1[$ such that for every $i = 1, \cdots, n_1$:

- $|\mu|(E_i - K_i) < \dfrac{\delta}{2n_1}$;
- $\delta[A(K_i\Sigma^2)] < +\infty$;
- for every $E \in K_i\Sigma^+$ there exists $F \in E\Sigma^+$ such that $|\mu|(F) > \alpha_i|\mu|(E)$ and $A(F, \varepsilon) \neq \emptyset$.

Then one easily proves **(RN.2b.2)** and **(RN.2b.3)** with $C = \bigcup_{i=1}^{n_1} K_i$, $r = \sum_{i=1}^{n_1} r_i$, $\alpha = \dfrac{1}{2n_1} \min\{\alpha_1, \cdots, \alpha_{n_1}\}$.

We now turn to the implication **(RN.2)** $\Rightarrow$ **(RN.1).** Let $(\delta_n)_n$ be a sequence of positive numbers decreasing to zero. To each $n \in I\!N$ there corresponds $C_n \in \Sigma$ such that $|\mu|(\Omega - C_n) < \delta_n$, and $A(C_n\Sigma^2)$ is Q_M-uniformly bounded. We take $E_1 = C_1, E_n = C_n - \cup_{i=1}^{n-1} C_i$, possibly eliminating the $|\mu|$-vanishing E_i's. The sequence thus obtained is a μ-exhaustion of Ω and $A(E_j\Sigma^2)$ is Q_M-uniformly bounded for each j. By Lemma 3.6 applied to each E_n one obtains the existence of a Q_M-uniformly bounded multifunction G_n which is the limit, uniformly with respect to $p \in Q_M$ and $x \in \Omega$, of Q_M-strongly μ-integrable multifunctions, and which vanishes outside E_n and such that

$$M(E_n \cap H) = \int_H G_n d\mu.$$

It is then possible to define the formal series $G(x) = \sum_n G_n(x)$. In fact the support of G_n is E_n and the sets E_n's are pairwise disjoint by construction. We now prove that G is μ-integrable. The set $A_\varepsilon^{(n)} = \{x \in \Omega : h_p(G(x), \sum_{k=1}^n G_k(x)) > \varepsilon\} \in \Sigma$ and it is a subset of $\Omega - \cup_{k=1}^n E_k$ and hence $|\mu|(A_\varepsilon^{(n)}) \to 0$ for every $p \in Q$, namely $\sum_{k=1}^n G_k(x)$ μ-converges to $G(x)$ uniformly with respect to $p \in Q_M$. We now prove that $\int h_p(\sum_{k=1}^n G_k(x), \{0\}) d|\mu| \ll \mu$ uniformly with respect to n and for every $p \in Q_M$: in fact the sequence $h_p(\sum_{k=1}^n G_k(x), \{0\}) = \sum_{k=1}^n h_p(G_k(x), \{0\})$ is non decreasing and

$$\int_E h_p(\sum_{k=1}^n G_k(x), \{0\}) d|\mu| \le \sum_{k=1}^n \int_E h_p(G_k(x), \{0\}) d|\mu| \le \sum_{k=1}^n |M|_p(E \cap E_k) \le |M|_p(E).$$

Therefore, by the Vitali Convergence Theorem, for every $p \in Q_M$

$$h_p\left(\int_E G d\mu, \int_E \sum_{k=1}^n G_k d\mu\right) \to 0$$

namely

$$h_p\left(\int_E G d\mu, M(E \cap (\cup_{k=1}^n E_k))\right) \to 0.$$

Therefore it is enough to show that

$$h_p\left(M(E), M(E \cap (\cup_{k=1}^n E_k))\right) \to 0.$$

$$h_p\left(M(E), M(E \cap (\cup_{k=1}^{n} E_k))\right) =$$

$$= [h_p(M(E - \cup_{k=1}^{n}(E \cap E_k)) + M(E \cap (\cup_{k=1}^{n} E_k)), M(E \cap (\cup_{k=1}^{n} E_k)))] =$$

$$= h_p(M(E - \cup_{k=1}^{n}(E \cap E_k)), \{0\})$$

and the assertion follows from the absolute continuity of M with respect to μ and the definition of $(E_n)_n$.

COROLLARY 3.12. *Let* $M : \Sigma \to Y$ *be a b.v. finitely additive multimeasure and* $\mu : \Sigma \to I\!R$ *a bounded finitely additive measure. Then the following are equivalent:*

(RN.1) *there exists a* μ*-integrable multifunction* $G : \Omega \to Y$ *satisfying:*

(RN.1a) *there is a* μ*-exhaustion* $(E_i)_i$ *of* Ω *such that for every* i $G : E_i \to Y$ *is the limit, uniformly with respect to* $p \in Q_M$ *and* $x \in E_i$, *of* Q_M*-strongly* μ*-integrable multifunctions;*

(RN.1b) *for every* i *there is* $r_i > 0$ *such that* $\sup_{p \in Q_M} h_p(G(x), \{0\}) \leq r_i$ *for all* $x \in E_i$;

(RN.1c) $\displaystyle\int_E G d\mu = M(E)$ *for every* $E \in \Sigma$;

(RN.2)

(RN.2a) $M \ll \mu$;

(RN.2b) *for every* $\varepsilon > 0$ *and* $\delta > 0$ *there are* $C \in \Sigma^+$ *and* $\alpha \in]0, 1[$ *such that:*

(RN.2b.1) $|\mu|(\Omega - C) < \delta$;

(RN.2b.2) $A(C\Sigma^2)$ *is* Q_M*-uniformly bounded;*

(RN.2b.3') *for every* $E \in C\Sigma^+$ *there is* $F \in E\Sigma^+$ *such that* $|\mu|(F) > \alpha|\mu|(E)$ *and* $\delta(A(F\Sigma^2)) < \varepsilon$.

REMARK 3.13. The absolute continuity of M with respect to μ as defined in 1.11 is in a strong sense: in [8] C. Castaing, A. Touzani, M. Valadier have introduced a weaker definition: M is absolutely continuous with respect to μ (and we write $M \ll_* \mu$) iff for every $x^* \in X^*$ the finitely additive measure $\delta^*(x^*|M(\cdot))$ is absolutely continuous with respect to μ. In [8] the following Theorem is proven:

THEOREM 3.14 ([8]). *Let* $\mu : \Sigma \to I\!R_0^+$ *be a bounded finitely additive measure and* $M : \Sigma \to Y$ *a finitely additive multimeasure with* $M \ll_* \mu$. *The following two conditions are equivalent:*

1. $\{M(A) : A \in \Sigma\}$ *is precompact in* Y*;*
2. *for every* $p \in Q$ *and* $\varepsilon > 0$ *there exists a simple multifunction* $F = \sum_{i=1}^n \dfrac{M(E_i)}{\mu(E_i)} 1_{E_i}$, *where* $(E_i)_i$ *is a finite decomposition of* Ω*, such that*

$$h_p\left(\int_A F d\mu, M(A)\right) \le \varepsilon \qquad \textit{for every} \quad A \in \Sigma.$$

In this case one obtains an approximated Radon-Nikodym derivative. When the density exists from Theorem 3.11 the absolute continuity of M with respect to μ is in the strong sense and, in this case, by Theorem 3.14, the existence of a density implies that the range of M is precompact. The converse is not true as shown by example 3.7 di [4]. In fact let μ_L and δ be the finitely additive measures of the example. The range of μ_L is precompact but the Radon-Nikodym derivative of μ_L with respect to $\mu_L + \delta$ does not exist. Observe that Theorem 3.14 obtains an approximated Radon-Nikodym derivative depending on $p \in Q$. On a following paper we shall face the study of this kind of integration by seminorms ([1]).

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