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A Radon-Nikodym theorem for the Bartle-Dunford-Schwartz integral with respect to finitely additive measures *

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1. Introduction

One the most recent development in vector integration is directed toward defining the integral in a locally convex space. This generalization is not artificial, but follows the current investigation concerning, for example, Stochastic Processes. The existence of a density is indeed a fundamental tool for the decomposition theorems that allow to single out the "good" integrators in the theory of Stochastic Integration. The

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setting of locally convex spaces thus makes it possible to develop a theory of Stochastic integration e.g. in nuclear spaces as the space of distributions ([2]).

The integral of a scalar function with respect to a vector finitely additive measure μ can be defined in several different ways (see [7]). In particular, one can consider the Bartle–Dunford–Schwartz integral defined in the following way: let X be a locally convex topological vector space, (Ω, Σ) a measurable space, $\mu : \Sigma \to X$ a finitely additive measure, f a scalar valued function such that for every $x^* \in X^*, f \in L^1(x^*\mu)$; then f is *integrable in the Bartle–Dunford–Schwartz sense* if for every $E \in \Sigma$ there exists $x_E \in X$ such that

$$x^*(x_E) = \int_E f dx^* \mu$$

for every $x^* \in X^*$. For this integral Musial[10] has given a Radon-Nikodym Theorem when μ is countably additive: he obtains the equivalence of the existence of a density with three equivalent conditions expressing the suitable *absolute continuity.*

The aim of this paper is to extend the Radon-Nikodym Theorem of [10] to the case of finitely additive measure's.

The Radon-Nikodym Theorem here proven makes use of the Moedomo-Uhl kind of assumption [9].

The first complication arises from the fact that the finite additivity, due to the lack of the Radon-Nikodym Theorem even in the scalar setting, does not guarantee under the simple assumption of the absolute continuity the existence of the scalar density $\dfrac{dx^*\nu}{dx^*\mu}$.

Hence one has to assume such existence or some conditions ensuring it, like those in [4]. Moreover, since the proof in the countably additive case is based upon the existence of a lifting, it cannot be mimicked in the present setting; its role in the proof is somehow replaced by assuming that μ admits a Rybakov control. This condition, which is not necessarily satisfied when X is a locally convex topological vector space even for s-bounded μ (as it is when X is a Banach space), is shortly discussed at the end of the paper.

2. Preliminaries

Troughout the sequel $\mathcal{X}$ will be a sequentially complete locally convex topological vector space. Let (Ω, Σ) be a measurable space.

(C1) Let $\nu, \mu : \Sigma \to \mathcal{X}$ be two finitely additive measures (f.a.m.'s) such that for every $x^* \in \mathcal{X}^*$ the f.a.m.'s $x^*\mu$ and $x^*\nu$ are b.v. Assume also that μ admits a Rybakov control $\lambda = |x_0^*\mu|$.

We begin with some definitions.

DEFINITION 1. We shall say that ν is *scalarly uniformly absolutely continuous* with respect to μ, and write $\nu \lll \mu$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x^* \in \mathcal{X}^*$ and $E \in \Sigma$ $|x^*\mu|(E) < \delta$ yields $|x^*\nu|(E) < \varepsilon$

DEFINITION 2. We shall say that ν is *scalarly dominated* by μ if there exists $M > 0$ such that $|x^*\nu|(E) \leq M|x^*\mu|(E)$ for every $E \in \Sigma$ and $x^* \in \mathcal{X}^*$.

DEFINITION 3. We shall say that ν is *subordinated* to μ if there exists $N > 0$ such that for every $E \in \Sigma$ $\nu(E) \in \overline{aco}\{\mu(F), F \in E \cap \Sigma\}$ where $aco(A) = \{\sum_{i=1}^{n} \alpha_i x_i, x_i \in A, \sum_{i=1}^{n} |\alpha_i| = 1\}$.

Let $\mathcal{P}$ be the family of seminorms generating the topology of $\mathcal{X}$.

We shall say that the range of μ $R(\mu)$ is *bounded* if for every $p \in \mathcal{P}$ there exists $\lambda_p > 0$ such that $R(\mu) \subset \lambda_p\{x \in \mathcal{X} : p(x) \leq 1\}$.

If $R(\mu)$ is bounded, we shall set

$$\mathcal{P}_\mu = \{\frac{p}{\lambda_p}, p \in \mathcal{P}\}$$

and

$$\mathcal{X}^*_{\mu,p} = \{x^* \in \mathcal{X}^* : x^* \leq p\}.$$

Let

$$G_{1,\mu} = \{f : \Omega \to I\!R : f \in L^1(|x^*\mu|) \forall x^* \in \bigcup_{p \in \mathcal{P}_\mu} \mathcal{X}^*_{\mu,p}\}.$$

DEFINITION 4. Let $f \in G_{1,\mu}$; we shall say that f is *μ-integrable* provided for every $A \in \Sigma$ there exists $v(A) \in \mathcal{X}$ such that

$$x^* v(A) = \int_A f d(x^*\mu)$$

for every $x^* \in \mathcal{X}^*$. Then we shall set $\int_A f d\mu = v(A)$.

We shall need in the sequel the following finitely additive extension of a classical theorem

THEOREM 1. **(Image Law)** *Let $m, s : \Sigma \to I\!R$ be two f.a.m.'s with bounded variation. If $s = \int f dm$ and $h : \Omega \to I\!R$ is a Σ-measurable bounded function, then $fh \in L^1(|m|)$ and*

$$\int h ds = \int h f dm \tag{1}$$

Proof. As h is bounded, there exists $M > 0$ such that $|h(\omega)| \leq M$ for every $\omega \in \Omega$ and thus $|hf| \leq M|f| \in L^1(|m|)$: hence the m-integrability of hf is straightforward. It remains to prove the equality in (1).
Let h be simple: then (1) is obvious. Assume now that $h \geq 0$; then the Lebesgue ladder trick gives a sequence $(h_n)_n$ of simple functions such that

$h_n \leq h_{n+1} \leq h$ for every $n \in I\!N$;

h_n converges uniformly to h.

By the m-integrability of f there exists a defining sequence of simple functions $(f_n)_n$ such that $f_n \to f$ in $L^1(|m|)$ ([7]). Let $s_n(\cdot) = \int_{(\cdot)} f_n dm$; then for every $\varepsilon > 0$ there exists $\overline{n} \in I\!N$ such that for all $n > \overline{n}$

$$|s_n - s|(E) = \int_E |f_n - f| d|m| \leq \int_\Omega |f_n - f| d|m| < \varepsilon$$

for every $E \in \Sigma$, namely $|s_n - s|(\cdot)$ converges to 0 uniformly in Σ.
Let $E \in \Sigma$ be fixed and let us put $a_{i,n}(E) = \int_E h_n ds_i$; then, since h_n and f_i are simple, $a_{i,n}(E) = \int_E h_n f_i dm$. We shall show that for every $i \in I\!N$ there exists $\lim\limits_{n\to\infty} a_{i,n}(E)$ and that $\lim\limits_{i\to\infty} a_{i,n}(E)$ exists uniformly in $n \in I\!N$. Then it will follow that

$$\lim_{i\to\infty}\lim_{n\to\infty} a_{i,n}(E) = \lim_{n\to\infty}\lim_{i\to\infty} a_{i,n}(E) = \lim_{i,n\to\infty} a_{i,n}(E).$$

Let $i \in I\!N$ be fixed; from the uniform convergence of h_n to h we have that h_n s_i-converges to h and $\int_E h_n ds_i \to \int_E h ds_i$ that is $\lim\limits_{n\to\infty} a_{i,n}(E)$ exists.
Moreover, if n is fixed, as h_n is simple, it is easy to check that

$$|a_{i,n}(E) - \int_E h_n ds| = |\int_E h_n ds_i - \int_E h_n ds| \leq M|s_i - s|(E)$$

and since, as observed, $|s_i - s|(\Omega) \to 0$ it follows that $\lim_{i\to\infty} a_{i,n}(E) = \int_E h_n ds$ uniformly with respect to n.

Then

$$\int_E hf dm = \lim_{i,n\to\infty} \int_E h_n f_i dm = \int_E h ds.$$

For general bounded h it is enough to decompose $h = h^+ - h^-$.

DEFINITION 5. We shall say that a measurable function $f : \Omega \to I\!\!R$ is λ-*null* if for every $\varepsilon > 0$ it is $\lambda(\{|f| > \varepsilon\}) = 0$.

Observe that if $f = 0$ λ-a.e. then f is λ-null, while the converse is true if λ is σ-additive, or at least λ fulfills the condition

(σ) *the ideal of λ-null sets is closed under countable unions.*

We list two straightforward properties of λ-null functions that we will need in the sequel:

(p.1) f is λ-null iff $\int_E |f| d\lambda = 0$ for every $E \in \Sigma$;

(p.2) if f is λ-null and g is bounded then fg is λ-null.

3. Radon-Nikodym Theorem

We shall now prove the main theorem.

THEOREM 2. **(Radon-Nikodym)** *Let (Ω, Σ) be a measurable space, and $\nu, \mu : \Sigma \to \mathcal{X}$ be two f.a.m.'s satisfying* **(C1)** .

Assume that for every $* \in X^*$ there exists a λ-exhaustion $(\Omega_n^{x^*})_n$ such that for every $n \in I\!\!N$ the set

$$S_n^{x^*} = \{\frac{x^*\mu(A)}{\lambda(A)}, A \in \Omega_n^{x^*} \cap \Sigma, \lambda(A) > 0\}$$

is bounded for every $x^* \in \mathcal{X}^*$*, and for every* $n \in I\!N$ *the ranges* $R(\mu|_{\Omega_n^{x^*} \cap \Sigma})$ *and* $R(\nu|_{\Omega_n^{x^*} \cap \Sigma})$ *are closed.*
Then the following are equivalent:

i) $\nu \lll \mu$;

ii) ν *is scalarly dominated by* μ*;*

iii) *there exists* $\vartheta : \Omega \to I\!R$ *bounded and such that*

$$x^*\nu(E) = \int_E \vartheta dx^*\mu$$

for every $E \in \Sigma$ *and for every* $x^* \in \mathcal{X}^*$*;*

iv) ν *is subordinated to* μ*.*

Proof. The proofs of the implications **i)** $\Longleftrightarrow$**ii)** and **iv)** $\Longrightarrow$ **ii)** are essentially the same as in [6]; also the implication **iii)** $\Longrightarrow$ **iv)** can be proven in the same fashion as in the same paper, by making use of the results in [7] that are the extensions in the finitely additive setting of those of Lewis [6]. Hence it only remains to prove the implication **ii)** $\Longrightarrow$ **iii)** .

Claim 1 - *For every* $x^* \in \mathcal{X}^*$ *there exist* $\dfrac{dx^*\mu_n}{d\lambda}$, $\dfrac{dx^*\nu_n}{d\lambda}$*, where* $\mu_n = \mu|_{\Omega_n^{x^*} \cap \Sigma}$ *and* $\nu_n = \nu|_{\Omega_n^{x^*} \cap \Sigma}$.

Proof. We shall show the assertion for $(x^*\mu, \lambda)$ since the proof for $(x^*\nu, \lambda)$ is analogous.
Let $x^* \in \mathcal{X}^*$ be fixed and let $n \in I\!N$. For every $a, b \in I\!R$ let $y^* = ax^* + bx_0^*$: since our assumptions ensure that $R(y^*\mu_n)$ is closed, the signed measure $y^*\mu_n$ admits a Hahn dcomposition. Then from [3] Lemma 4.3 there

exists $\dfrac{dx^*\mu_n}{d|x_0^*\mu|} = \dfrac{dx^*\mu_n}{d\lambda}$.
We shall denote $f_{x^*}^{(n)} = \dfrac{dx^*\nu_n}{d\lambda}$ and $g_{x^*}^{(n)} = \dfrac{dx^*\mu_n}{d\lambda}$.
Observe also that, without loss of generality, one can choose a representative of $g_{x_0^*}^{(n)}$ such that the set $\{g_{x_0^*}^{(n)} = 0\}$ is empty, and $|g_{{x^*}_0}^{(n)}| = 1$.

Claim 2 - *$g_{x^*}^{(n)}$ is essentially bounded on $\Omega_n^{x^*}$ for every $n \in I\!N$.*

This follows immediately from the boundedness of $S_n^{x^*}$.
Then define $L_{x^*}^{(n)} = supess|g_{x^*}^{(n)}|$ and let $\Omega_{0,x^*}^{(n)} \subset \Omega_n^{x^*}$ be such that $\lambda(\Omega_{0,x^*}^{(n)}) = 0$ and

$$|g_{x^*}^{(n)}(\omega)| \leq L_{x^*}^{(n)}$$

for every $\omega \notin \Omega_{0,x^*}^{(n)}$. Since $(\Omega_n^{x^*})_n$ is a λ-exhaustion, $\lambda(\bigcup_n \Omega_{0,x^*}^{(n)}) = 0$.
Define $\Omega'_{n,x^*} = \Omega_n^{x^*} - \Omega_{0,x^*}^{(n)}$; then $\lambda(\bigcup_n \Omega'_{n,x^*}) = \lambda(\Omega)$ and $(\Omega'_{n,x^*})_n$ is a λ-exhaustion of Ω. Since the $\Omega_n^{x^*}$'s are pairwise disjoint, also the Ω'_{n,x^*}'s are pairwise disjoint, so we can define $g_{x^*} = \sum_{n=1}^{\infty} g_{x^*}^{(n)} 1_{\Omega'_{n,x^*}}$ and $f_{x^*} = \sum_{n=1}^{\infty} f_{x^*}^{(n)} 1_{\Omega'_{n,x^*}}$.
Let $x^* \in \mathcal{X}^*$ be fixed and define $H_{x^*} = \{g_{x^*} \neq 0\}$.

Claim 3 - *The function $\dfrac{f_{x^*}}{g_{x^*}} - \dfrac{f_{{x^*}_0}}{g_{{x^*}_0}}$ is λ-null in $H_{x^*} \cap \Omega'_{n,x^*}$.*

Proof. From the linearity of the maps $x^* \to f_{x^*}$ and $x^* \to g_{x^*}$, and from assumption **ii)** for every $E \in \Sigma$ and for every $\beta_1, \beta_2 \in I\!R$

$$\int_E |\beta_1 f_{x^*} + \beta_2 f_{x_0^*}| d\lambda \leq M \int_E |\beta_1 g_{x^*} + \beta_2 g_{x_0^*}| d\lambda \qquad (2)$$

Let n be fixed and let $\widetilde{\Omega}_{n,x^*} = \Omega'_{n,x^*} \cap H_{x^*}$. Then g_{x^*} is bounded in $\widetilde{\Omega}_{n,x^*}$ and therefore there exists a sequence of simple functions $(\gamma_{x^*,k}^{(n)})_k$

that converges uniformly to $g^{(n)}_{x^*}$ in $\widetilde{\Omega}_{n,x^*}$; also, since in $\widetilde{\Omega}_{n,x^*}$ $g_{x^*} \neq 0$ for k large enough $\gamma^{(n)}_{x^*,k} \neq 0$ in $\widetilde{\Omega}_{n,x^*}$.

Since $\gamma^{(n)}_{x^*,k}, g_{x^*_0}$ are simple functions, it is possible to decompose $\widetilde{\Omega}_{n,x^*}$ into finitely many subsets where both these functions are constant. Let $(E^{(j)}_n)^{r(x^*,k)}_{j=1}$ be such a decomposition. Taking $\beta_1 = \dfrac{1}{\gamma^{(n)}_{x^*,k}(E^{(j)}_n)}$ and $\beta_2 = -\dfrac{1}{g_{x^*_0}(E^{(j)}_n)}$ in (2) we then find

$$\int_{E\cap E^{(j)}_n} |\frac{f_{x^*}}{\gamma^{(n)}_{x^*,k}} - \frac{f_{x^*{}_0}}{g_{x^*{}_0}}|d\lambda \leq M \int_{E\cap E^{(j)}_n} |\frac{g_{x^*}}{\gamma^{(n)}_{x^*,k}} - 1|d\lambda$$

for every j and since the $E^{(j)}_n$'s are finitely many, for every $E \subset \widetilde{\Omega}_{n,x^*}$

$$\int_{E} |\frac{f_{x^*}}{\gamma^{(n)}_{x^*,k}} - \frac{f_{x^*{}_0}}{g_{x^*{}_0}}|d\lambda \leq M \int_{E} |\frac{g_{x^*}}{\gamma^{(n)}_{x^*,k}} - 1|d\lambda \qquad (3)$$

By taking the limit for $k \to \infty$ in (3) we then obtain

$$\int_{E} |\frac{f_{x^*}}{g_{x^*}} - \frac{f_{x^*{}_0}}{g_{x^*{}_0}}|d\lambda = 0$$

for every $E \subset \widetilde{\Omega}_{n,x^*}$. Then **(p1)** implies **Claim 3.**

Let $\vartheta = \dfrac{f_{x^*{}_0}}{g_{x^*{}_0}}$. From **(p. 2)** the function $f_{x^*} - \vartheta g_{x^*}$ is λ-null in $\widetilde{\Omega}_{n,x^*}$.

Claim 4 - *For every* $\varepsilon > 0$ $|f_{x^*}| \leq (M+\varepsilon)|g_{x^*}|$ *λ-a.e. in* $\Omega^{x^*}_n$, *where* M *is that of the scalar domination.*

Proof. Indeed from the assumption **ii)** for every $E \in \Omega^{x^*}_n \cap \Sigma$

$$\int_E |f_{x^*}|d\lambda \leq M \int_E |g_{x^*}|d\lambda.$$

Hence one can easily prove that for every $\varepsilon > 0$

$$|f_{x^*}| < (M+\varepsilon)|g_{x^*}|$$

λ-a.e. in $\Omega_n^{x^*}$.

From **Claim 4** it follows that if $\omega \in {\Omega'}_{nx^*} - H_{x^*}$, namely if $g_{x^*}(\omega) = 0$ then λ-a.s. $f_{x^*}(\omega) = 0$; therefore $f_{x^*} = 0$ λ-a.e. in ${\Omega'}_{nx^*} - H_{x^*}$ and then $f_{x^*} - \vartheta g_{x^*} = 0$ λ-a.e. in ${\Omega'}_{nx^*} - H_{x^*}$.

We can therefore conclude that $f_{x^*} - \vartheta g_{x^*}$ is λ-null in ${\Omega'}_{nx^*}$.

Claim 5 - *For every $x^* \in \mathcal{X}^*$, $x^*\nu(E) = \int_E \vartheta dx^*\mu$.*

Proof. From **(p.1)** we find

$$|\int_E f_{x^*} d\lambda - \int_E \vartheta g_{x^*} d\lambda| \leq \int_E |f_{x^*} - \vartheta g_{x^*}| d\lambda = 0$$

for every $E \subset {\Omega'}_{n,x^*}$ and thus for every $E \subset {\Omega'}_{n,x^*}$

$$\int_E f_{x^*} d\lambda = \int_E \vartheta g_{x^*} d\lambda.$$

Observe also that, since $|g_{{x^*}_0}| = 1$ from **Claim 4** $|f_{{x^*}_0}| < M + \varepsilon$ λ-a.e. in each ${\Omega'}_{n,x^*}$ whence ϑ is λ -a.e. bounded. From Theorem 1, for every $E \in \Sigma$ and for $x^* \in \mathcal{X}^*$ fixed

$$x^*\nu(E \cap \Omega_n^{x^*}) = \int_{E \cap \Omega_n^{x^*}} f_{x^*} d\lambda = \int_{E \cap \Omega_n^{x^*}} \vartheta g_{x^*} d\lambda = \int_{E \cap \Omega_n^{x^*}} \vartheta dx^*\mu.$$

Since $(\Omega_n^{x^*})_n$ is a λ-exhaustion of Ω, for every $E \in \Sigma$

$$x^*\nu(E) = \sum_{i=1}^k x^*\nu(E \cap \Omega_i^{x^*}) + x^*\nu[E \cap (\bigcup_{i=k+1}^\infty \Omega_i^{x^*})] =$$
$$\sum_{i=1}^k \int_{E \cap \Omega_i^{x^*}} \vartheta dx^*\mu + x^*\nu[E \cap (\bigcup_{i=k+1}^\infty \Omega_i^{x^*})],$$

while

$$\int_E \vartheta dx^*\mu = \int_{E \cap (\bigcup_{i=1}^k \Omega_i^{x^*})} \vartheta dx^*\mu + \int_{E \cap (\bigcup_{i=k+1}^\infty \Omega_i^{x^*})} \vartheta dx^*\mu$$

whence

$$|x^*\nu(E) - \int_E \vartheta dx^*\mu| \leq |\int_{E\cap(\bigcup_{i=k+1}^{\infty}\Omega_i^{x^*})} \vartheta dx^*\mu| + |x^*\nu[E\cap(\bigcup_{i=k+1}^{\infty}\Omega_i^{x^*})]|.$$

Since $\lim_{k\to\infty}\lambda[E\cap(\bigcup_{i=k+1}^{\infty}\Omega_i^{x^*})] = 0$ and $|x^*\nu| \ll \lambda$ and $|x^*\mu| \ll \lambda$, it follows that

$$x^*\nu(E) = \int_E \vartheta dx^*\mu$$

for every $E \subset \Omega$.

REMARK 1.

1. In [5] Drewnovski studied the existence of a Rybakov control for an $\mathcal{X}$-valued countably additive measure. He showed that in general a Rybakov control does not exist in l.c.t.v.spaces unless some further conditions are satisfied. He also gave a quite strong condition for a f.a.m. to admit a Rybakov control.

2. It is easy to mimick the previous proof provided μ admits a control λ such that for some $x^* \in \mathcal{X}^*$ $\lambda(\{\frac{dx^*\mu}{d\lambda} = 0\} = 0$. It could therefore be of interest to investigate whether a f.a.m. μ admitting a control λ always fulfills this condition.

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