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A survey on multivalued integration[1]

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Abstract

In this paper the Aumann and the Debreu integrals are introduced and results about their comparison are given in countably and finitely additive setting.

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1 Introduction

Robert J. Aumann was the first who introduced in 1965, in the paper *"Integrals of set valued functions"*, the definition of a set valued integral with respect to the Lebesgue measure λ. In his work he considered multifunctions $F : [0,1] \to 2^{I\!R^n} \setminus \emptyset$ and defined the integrals in terms of integrable selections in the following way:

$$\int F d\lambda = \left\{ \int f d\lambda, \text{ where } f \in L^1(\lambda) \text{ and } f(t) \in F(t) \text{ for almost every } t \in [0,1] \right\}.$$

The Aumann integral, that we denote from now on by (A)-integral, obviously always exists; nevertheless it can be empty. In any case this formulation, which is very

natural and useful in view of applications (as in Control Theory or Mathematical Economics), lacks in general a lot of properties. For instance: convexity, closedness, compactness and convergence theorems, which are very important in the mentioned applications.
So after the pioneer work of Aumann the problem of multivalued integration was extensively studied. I will mention here only the multivalued integration introduced by Gerard Debreu in [8]. The Debreu integral is an extension of the Bochner integral to the case of set valued functions. In his important paper Debreu also compared his integral with the (A)-integral and showed that the two integrals coincide when the multifunctions take values in the hyperspace of compact convex subsets of $I\!R^n$ with the Hausdorff distance h. In this way what is true for the (D)-integral is true for the (A)-integral.
Following this idea the multivalued integration has been extended to the case of infinite dimensional spaces either in countably additive or in finitely additive framework. The aim of this paper is to produce a short history about the "equivalent" definition of the Debreu integral.

2 Notations

We first introduce some notations and obviously we define the (A)-integral in the infinite dimensional case.

Let (Ω, Σ) be a measurable space, $\mu : \Sigma \to I\!R_0^+$ a bounded countably (or finitely) additive measure and X a separable metric space. If A and B are bounded subsets of X, the Hausdorff distance between A and B is given by: $h(A, B) = \max\{e(A, B), e(B, A)\}$ where e denotes the excess and it is defined by $e(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|$.
We recall the notation of the following hyperspaces:

$$\begin{aligned} ck(X) &= \{C \subset X : C \neq \emptyset \text{ is compact and convex } \} \\ cwk(X) &= \{C \subset X : C \neq \emptyset \text{ is w-compact and convex } \} \end{aligned}$$

$$cfb(X) \quad = \quad \{C \subset X : C \neq \emptyset \text{ is bounded, closed and convex } \}$$

We remind that, in order to be sure that the (A)-integral admits measurable selections we have to require both the separability of the space and the "measurability" of the multifunction. The one that we introduce here is the classical Effros-measurability, but there are several others that can be found in [13] or in [7] or in [12]. $F : \Omega \to 2^X \setminus \{\emptyset\}$ is *Effros measurable* (or simply measurable) if for every open set U

$$F^{-}U = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma.$$

The integrable boundedness was introduced to ensure that the family of integrable selections, $S^1_F = \{f \in L^1_\mu(X) : f(\omega) \in F(\omega) \text{ a.e. } \}$ is non empty. A multifunction F is called *integrably bounded* if there exists a scalar function $g \in L^1_\mu(I\!R^+_0\ )$ such that $h(F(\omega), \{0\}) \leq g(\omega)$. In this case

$$(A) - \int_\Omega F d\mu = \left\{ \int_\Omega f d\mu, f \in S^1_F \right\} \neq \emptyset.$$

3 The Debreu Integral in countably additive case.

In his paper "Integration of Correspondences", Debreu introduced a new version for the set valued integral; he considered multifunctions with values in $ck(I\!R^n)$, proved that the family of simple multifunctions is dense in the space $\mathcal{L}^1(ck(I\!R^n))$ of measurable multifunctions which admit non empty (A)-integral, and defined the (D)-integral via simple multifunctions. Namely, a measurable multifunction $F : [0,1] \to ck(I\!R^n)$ is (D)*-integrable* if there exists a sequence of measurable simple multifunctions $(F_k)_k$ such that

$(D)_1$ $h(F_k, F)$ converges to zero λ a.e. (this is labelled throughout this paper as the *total measurability* of F);

$(D)_2$ $\lim_{m,k\to\infty} \int h(F_k, F_m) d\lambda = 0.$

In this case, since $(ck(I\!R^n), h)$ is complete (and separable), the (D)-integral of F is defined as follows:

$$(D) - \int F d\lambda = \lim_{k\to\infty} \int F_k d\lambda.$$

The importance of the result of Debreu lies in the fact that this integral is compact convex valued, and all the results known for the Bochner integral also hold for it. What's more, Debreu proved that for measurable and integrably bounded multifunctions the (A)-integral and the (D)-integral coincide. This result was extended later by several authors in the framework of a measurable space (Ω, Σ, μ) more general than $[0, 1]$ with the Lebesgue measure. I mention here, for the countably additive case, the contributions of [12], [6], [24], [17].

The results in [12] and [6] extend those of Debreu to the case of infinite dimensional spaces, when the measure μ is finite and atomless, and the space X is a separable Banach space, using the Radström embedding Theorem, [21]. We recall it for the sake of completeness. Let X be a metric space and let (M, h) be a family of elements of $(cfb(X), h)$ satisfying:

$(R)_1$ *M is closed with respect to the sum $+$ and with respect to the multiplication by a scalar number $\lambda \in I\!R_0^+$;*

$(R)_2$ *if $A \in M$ then $A + B_X$ is closed, where B_X denotes the closed unitary ball of X.*

then M can be embedded as a convex cone in a separable metric space N (if X is a Banach (reflexive) space we can choose N Banach (reflexive) too) in such a way that:

- *the embedding is isometric;*
- *the sum on N induces the same operation on M;*
- *the multiplication by a non negative number on N induces the same operation on M.*

In particular, $(ck(X), h), (cwk(X), h)$ satisfy the conditions $(R)_1, (R)_2$ and so measurable multifunctions with values in these hyperspaces can be viewed as single

valued functions. If X is reflexive the same is true for $(cfb(X), h)$.
In this way the definition of (D)-integral given in $(D)_1$ and $(D)_2$ can be easily extended to the infinite dimensional case. The difference between the results in [12] and those in [6] is due to the fact that $(ck(X), h)$ is separable and complete, while $(cwk(X), h)$ needs not to be separable, in general. Therefore Byrne had to assume the total measurability of the multifunction explicitly.
Finally for what concerns the equivalence between (A)-integral and (D)-integral the results obtained are the following:

Theorem 3.1 (Hiai–Umegaki [12], Theorem 4.5) *Let X be a separable metric space. If $F : \Omega \to ck(X)$ is (A)-integrable then $cl\int_\Omega F d\mu$ (moreover if X has the RNP, $\int_\Omega F d\mu$ itself) is equal to the Bochner integral of F considered as a function in $L^1_\mu(N)$.*
If X is reflexive and $F : \Omega \to cfb(X)$ is (A)-integrable, then $\int_\Omega F d\mu$ is equal to the Bochner integral of F considered as a function in $L^1_\mu(N)$.

Observe that in [12] the (D)-integral is called Bochner integral. Moreover if X is complete, the graph-measurability is equivalent to the Effros measurability and then the first result of the previous theorem holds simply for the (A)-integral, (see for example Theorem 5.23 of [14]). We also report Byrne's Theorem:

Theorem 3.2 (Byrne [6]) *If $F : \Omega \to cwk(X)$ is measurable, totally measurable and integrably bounded then its (A)-integral and (D)-integral coincide.*

In order to prove the equivalence of integrals, Byrne observed that the proof given by Debreu is valid, with some modification, only if the space X is reflexive. Extending the proof to a more general case requires sufficient conditions for the weak compactness of $S^1_F \subset L^1_\mu(X)$. In particular Byrne proved that if F is measurable, integrably bounded, with values in $cwk(X)$ and totally measurable, then the set $S^1_F \cup \{\cup_n S^1_{F_n}\}$ is relatively weakly compact in $L^1_\mu(X)$.

To extend now the comparison to the target hyperspace $M = cfb(X)$ with the Hausdorff distance, note first that it can be embedded as a convex cone in a suitable Banach space N, if we replace the sum $+$ with the operation $\dot{+}$, defined as $A \dot{+} B = \text{cl}(A+B)$, (see [10]).
In this way a multifunction with values in $cfb(X)$ can be viewed as a N-valued function, and it is possible to define its (D)-integral.
However, to compare it with its (A)-integral, it is not possible to follow the proof of Byrne, since it is strictly related to the Grothendieck Lemma and to the weak compactness of the values of the multifunctions.
In [24] I obtained the equivalence under the further hypothesis that X^* has the RNP. In this case in fact it is possible to use some results concerning the weak compactness of subsets of $L^1_\mu(X)$ (see [9]) and to prove that S^1_F is weakly relatively compact if F is measurable and integrably bounded.
Moreover if F is totally measurable it is possible to construct an integrably bounded, measurable $cfb(X)$-valued multifunction G and a sequence of simple measurable multifunction $(G_n)_n$ such that the sequence satisfies $(D)_1$ and $(D)_2$ and $G_n \subset G$ for every $n \in \mathbb{N}$. Using this result it is possible to obtain the following

Theorem 3.3 [[24], Theorem 3.11] *Suppose that X is a separable Banach space and that its dual X^* has RNP. If $F : \Omega \to cfb(X)$ is a totally measurable and integrably bounded multifunction, then*

$$(A) - \int F d\mu = (D) - \int F d\mu.$$

An analogous result appears in [17]. There the comparison of the two integrals is obtained when X is a separable reflexive Banach space and the multifunctions take values in the family of sets in $cfb(X)$ which have the so called Drop property. The authors modified the definition of (D)-integral asking that the defining sequence of simple multifunctions is equibounded and converges to F is the sense of Mosco, and prove that if F is (D)-integrable in this sense and admits integrable selections, then again the two integrals coincide.

4 The Multivalued Integral in finitely additive case.

The importance of integrating correspondences with values in infinite dimensional commodities spaces was recognized long time ago. Quite recently attention has been posed to the case of finitely additive measures (see for example the paper of T.E. Armstrong and M.K. Richter [1]).
In fact, if we consider economies with negligible individuals, we have to ask that the measure μ is atomless. If we denote by Σ the σ-algebra of all possible coalitions of agents there are no economic reasons to exclude a priori some coalitions, but we have to take under consideration the phenomenon concerning non atomicity:

Theorem 4.1 [Ulam's Theorem (1930)] *If we assume CH and we suppose that $|\Omega| > \aleph_0$ the only atomless countably additive measure $\mu : 2^{\Omega} \to \mathbb{R}_0^+$ is the null measure.*

Therefore, it is clear that if we want that Σ coincides with the power set 2^{Ω} we cannot assume the non atomicity of μ if we assume CH. On the other hand, in the finitely additive case, and in view of the mentioned applications in Mathematical Economics, all the mentioned features are compatible, and monotonicity can be strengthened: the usual alternative is the strong continuity of μ, namely μ is *strongly continuous* if and only if for every $\varepsilon > 0$ there exists a finite decomposition of Ω, $A_i \in \Sigma, i = 1, \ldots, n$ such that $\mu(A_i) \leq \varepsilon$ for every $i = 1, \ldots, n$.
Indeed the following result is known:

Theorem 4.2 [Nachbin Theorem (1965)] *Every atomless measure $\mu : \Sigma \to \mathbb{R}_0^+$ can be extended to the power set in such a way that the extension is finitely additive and strongly continuous.*

This theorem allows the extension to the whole power set. On the other side, in the countably additive case, everything agrees with what is already known, thanks to the following theorem:

Theorem 4.3 [Saks Theorem (1934)] *If μ is countably additive then it is atomless if and only if is it strongly continuous.*

Achille Basile in [3], was the first who studied and applied (A)-integration with respect to quasi-countably additive measures μ. The multivalued integration was also studied by the author jointly with Anna Martellotti in two different directions. A multivalued integration in the sense of Debreu was developed in [18] and [22, 23] when the multifunctions take values in a complete subspace of all closed convex and bounded subsets of a locally convex topological vector space X. In the first case the defining sequence of simple multifunctions does not depend on the family of seminorms. In both papers some properties for the integral are obtained, like convergence theorems and Radon-Nikodym theorems.

In [19, 20] the authors introduce the (A)-integral in the finitely additive setting and compared it with the (D)-integral given in [18]. The first problem was to decide what kind of measurability and integrability to assume in order to ensure the existence of "good" selections. For "good" selections of a multifunction F we mean that with them the (A)-integral is non empty and that it is possible to reobtain it in the countably additive case as a particular case. For this reason in these papers Σ is an arbitrary σ-algebra instead of the power set, even if the power set eliminates all problems concerning measurability. Indeed the Ulam Theorem does not permit to consider the countably additive case as a corollary of the finitely additive case.

Several results about the existence of selections in the countably additive case can be found in the literature. We quote here those of K. Kuratowski and C. Ryll Nardewski [15] and those of C. J. Himmelberg [13]. On the contrary, before [19, 20], there were no existence results when μ is only finitely additive.
In [19, 20] two different theorems for the existence of selections, one for multifunctions with values in $ck(X)$ and the other for multifunctions ranging in $cfb(X)$ are obtained.

Proposition 4.4 [19] *Let X be a separable Banach space. Let $F : \Omega \to ck(X)$ be a* *measurable multifunction such that* $cl\{F(\Omega)\} = cl\{\bigcup_{\omega\in\Omega} F(\omega)\}$ *is a compact subset of* X*; then* $S_F \neq \emptyset$*, where* S_F *denotes the family of selections of* F *which are totally measurable.*

When the compactness of the range of F is dropped, the total measurability of the selections in not ensured any more. One can only obtain scalarly μ-measurable selections. For this reason, in [20], the authors considered $S^1_{F,P}$; the space of Pettis integrable selections.

Proposition 4.5 [20] *Let* X *be a separable reflexive Banach space. If* $F : \Omega \to cfb(X)$ *is measurable and integrably bounded, then* $S^1_{F,P} \neq \emptyset$*.*

In this way, if F is integrably bounded we can define the (A)-integral as the set of the integrals of its "integrable" selections, where we mean Bochner integrability in the compact convex case, Pettis integrability in the other one.
Moreover, in both cases one can define the (D)-integral via simple multifunctions, provided $(D)_1$ is replaced with

$(D)'_1$ $h(F_n, F)$ μ-converges to zero

which is the standard assumption in the finitely additive case. Indeed, in the finitely additive case almost everywhere convergence does not compare with convergence in μ-measure, and only this last one yields Convergence Theorems of the integrals (see for example [5, 16]).
Now the two ingredients are defined and we are ready to compare them, by making use of the results of [12, 6] for the countably additive case.
The bridge to move from the finitely additive setting to the countably additive one is represented by the Stone Theorem and the Stone transform of a measure and of a function. Here we suppose that (Ω, Σ) is complete with respect to μ (namely Σ contains all subsets of μ-null sets), we denote with S the Stone space associated to (Ω, Σ, μ), and with $\overline{\mu}$ the Stone extension of μ.
It was immediate to define the Stone extensions of selections and multifunctions

which are totally measurable following the idea of Feffermann [11] and taking into account that the hyperspace is embedded in a suitable Banach space N and that the multifunctions become functions with values in this space N; as it has been done in [19].

The situation was different and harder in the other case [20]. There a suitable definition for the extension of the Pettis integrable selections has been given, assuming the reflexivity of X. Given a scalarly μ-measurable selection f of F, we have introduced a new function ϕ, which is the *Stone extension* of f, which is defined $\overline{\mu}$-almost everywhere by: $\phi(s)(x^*) = \overline{x^* f}(s)$. We observe that, actually, if f is totally measurable, the extension obtained in this way coincides $\overline{\mu}$-a.e. with that one obtained via simple functions.

In both cases the following results hold:

- $(D) - \int_S \overline{F} d\overline{\mu} = (D) - \int_\Omega F d\mu$, where $\overline{F}$ is the extension of F;
- if $f \in S^1_F$ (in the sense of Bochner or Pettis) then its Stone extension $\overline{f} \in S^1_{\overline{F}}$ (which is always in the Bochner sense) and

$$\int_\Omega f d\mu = \int_S \overline{f} d\overline{\mu}.$$

Thanks to this fact and to the equivalence in the countably additive case, if F is (D)-integrable, immediately we obtain a first inclusion:

$$(A) - \int_\Omega F d\mu \subseteq (A) - \int_S \overline{F} d\overline{\mu} = (D) - \int_S \overline{F} d\overline{\mu} = (D) - \int_\Omega F d\mu.$$

To reverse this inclusion a further condition is needed. It is well known ([4, 16]) that when μ is finitely additive, $L^1_\mu(X)$ not to be complete. Therefore, in both papers, the completeness of $L^1_\mu(X)$ is explicitly assumed: in this way, every integrable selection φ of $\overline{F}$ is related to a integrable function f "close" to F in the following sense: $f \in L^1_\mu(X)$, $\overline{f} = \varphi$ and, for every $\alpha > 0$, $\mu(\{\omega \in \Omega : d(f(\omega), F(\omega)) \geq \alpha\}) = 0$ (which, in turn, reduces to $f \in S^1_F$ in the countably additive case).

In $ck(X)$ this is enough to reverse the inclusion thanks to the properties of the

Bochner integral, while in $cfb(X)$, where we suppose that the selections are Pettis integrable, only the closure of the (A)-integral coincides with the (D)-integral and we have to prove directly, when X is reflexive, that the (A)-integral is closed.
So, the two main results are:

Theorem 4.6 [19] *Let μ be a bounded finitely additive measure with $L^1_\mu(X)$ complete and suppose that X is a separable Banach space. If $F : \Omega \to ck(X)$ is (D)-integrable then,*

$$(A) - \int F d\mu = (D) - \int F d\mu.$$

Theorem 4.7 [20] *Let X be a separable reflexive Banach space, (Ω, Σ) a measurable space, μ a bounded finitely additive measure such that Σ is μ-complete and $L^1_\mu(X)$ is complete. If $F : \Omega \to cfb(X)$ is (D)-integrable, then*

$$(A) - \int F d\mu = (D) - \int F d\mu.$$

It should be mentioned that, in view of applications in Mathematical Economics, the convexity and the boundedness of the values of F are quite severe restrictions. However it appears impossible to keep this comparison technique in the non convex case: indeed in this case, examples show that the (D)-integral is not well defined. Even dropping the boundedness of the values appears difficult.
Some preliminary results without convexity and boundedness directly for the (A)-integral (that is, without comparison with the (D)-integral) have been obtained jointly with Anna Martellotti for some classes of multifunctions.

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